Measuring the power of parties within government coalitions

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Abstract

The paper presents a coalition-structure value that is meant to capture outside options of players in a cooperative game. It deviates from the Aumann-Drèze value by violating the null-player axiom. We use this value as a power index and apply it to the elections for the Berlin house of representatives in 2001.

Keywords: Power, government coalition, outside option, Aumann-Drèze value, Shapley value, null-player axiom.

JEL classification: C71, H1

1. Introduction

In 2001, elections for the Berlin house of representatives took place. As a result, a government coalition consisting of social democrates (SPD) and socialists (PDS) was formed. As usual, the fight for influence (and senator posts) between these
two parties and others was keenly followed by the general public. The SPD got the mayor’s office and 5 senator posts and the PDS got 3 senator posts, i.e. the ratio of posts is 6 to 3. Judging from a local newspaper, the Berlin socialists under the leadership of Mr. Gysi were disappointed with this result. The ”Berliner Morgenpost” notes (author’s translation): “In actual fact, the ratio of posts is 5.5 to 3.5, as Gysi bravely declared. For the SPD has to find agreement with the PDS on the candidate for the justice department.”

Gysi had hoped for 4 senator posts for his party. He backed this claim by a proportionality norm, pointing to 22% of the votes for his party versus 29% of the votes for the SPD. Indeed, 22/(22+29) is very close to 4/(4+5). From the point of view of political science, Mr. Gysi should have referred to seats instead of votes and, perhaps, taken recourse to more subtle power indices. However, the question of how to best predict political power is far from resolved. By and large, the predictive power of the proportionality norm seems to be higher than the predictive power of bargaining (in terms of the bargaining set), as shown by Laver & Schofield (1985). However, they try to identify characteristics of political-party systems which indicate whether proportionality or power is the best predictor in each special case. In this paper, we argue for a new power index that could help political parties to see whether they got their expected or ”fair” share of posts.

Perhaps the most obvious power index that comes to mind is the Shapley-Shubik index (see Shapley & Shubik (1954)). It would attribute a positive value to every party that is necessary to at least one government coalition. That is, the Shapley-Shubik index is an ex-ante value and does not help in determining the power of parties within alternative government coalitions. We are interested in an ex-post value. For that purpose, we will group parties that could form a government and then attribute the power to these parties, exclusively. Formally, we seek to develop a partitional value.

The reader will note that our players are parties as a whole. We rule out the possibility that members of the house of representatives defect and vote against their party affiliation. This is the usual assumption in the literature, with the notable early exception of Luce & Rogow (1956).

The value we are about to present cannot only be applied to simple games (used in political science for power measurement) but to any coalitional game (game in characteristic-function form). Therefore, the theoretical discussion of the values is not restricted to simple games. Of course, simple games suffice for the application to the Berlin election in 2001.

Building on the Shapley value, several partitional values (or values for coali-
tion structures) have been presented in the literature, most notably by Aumann & Drèze (1974) and Owen (1977). A partition (or coalition structure) divides the players into disjoint subsets. These subsets making up the partition are also called components. (Formal definitions are presented in the next section.) While Aumann and Drèze assume component efficiency (the players in a component share the worth of this component) Owen sticks to overall efficiency (as does the Shapley value). This difference is not merely technical but points to the question of what the components of the coalition structures "do". Are players organized in (active) components in order to do business together? Then the players within each component should arguably get its worth, as in the Aumann-Dreze value (AD-value). Or do players form bargaining components (unions etc.) that offer the service of all their members or no service at all - the Owen case.

By component efficiency, the AD-value seems a good candidate for an ex-post value measuring political power. However, differing outside options of players within a component do not bear on the payoff. This is not very realistic for power indices. Imagine two parties A and B within one government coalition. Party A could form alternative governments without B’s presence while party B does not have any such outside option. Then we should expect party A’s power to be larger than party B’s. The purpose of this paper is to incorporate outside options into the AD-value and apply this outside-option value to the Berlin election of 2001.

Let us look at an asymmetric version of an example used by Aumann & Myerson (1988). Assume player set $N = \{1, 2, 3\}$ and the characteristic, or coalition, function $v$ on $N$ which ascribes worths $v(\{1, 2\}) = v(\{2, 3\}) = 60$, $v(N) = 72$ and vanishes elsewhere. The Shapley value of this game is $(14, 44, 14)$. By efficiency, the whole of $v(N)$ is distributed among the players, but player 2 gets the lion’s share. Now let $\mathcal{P} = \{\{1, 2\}, \{3\}\}$ be a coalition structure. For this coalition structure, the AD-value $\varphi^{AD}$ and the outside-option value $\varphi^{oo}$ are

\[
\varphi^{AD}(v, \mathcal{P}) = (30, 30, 0),
\]

\[
\varphi^{oo}(v, \mathcal{P}) = (20, 40, 0).
\]

Both values are component-efficient. The outside-option value attributes a higher payoff to player 2 than to player 1 thus reflecting the outside opportunities of player 2 ($v(2, 3) = 60 > 0 = v(1, 3)$).

In spirit, the bargaining set is close to our value. (In the above example, the bargaining set yields $(0, 60, 0)$, a somewhat ”extreme” solution.) In fact, I find Maschler’s (1992, pp. 595) introducing remarks pertinent to the value presented in this paper:
During the course of negotiations there comes a moment when a certain coalition structure is "crystallized". The players will no longer listen to "outsiders", yet each [component] has still to adjust the final share of proceeds. (This decision may depend on options outside the [component], even though the chances of defection are slim).

The reader is also referred to the paper by Aumann & Drèze (1974) where outside options are modeled by redefining the characteristic function in a specific way. The result is a set of characteristic functions \( v^C_x \), one for each component \( C \) of a coalition structure \( \mathcal{P} \) (that also depends on a payoff vector \( x \in \mathbb{R}^n \)). The authors then go on to present coalition-structure spin-offs for the most widely used concepts within cooperative game theory. Interestingly, the coalition-structure Shapley value defined in that article (the AD-value introduced above) is the only one where the coalition functions \( v^C_x \) are not made use of.

What excuse could be offered for adding other outside-option values to those existent in the literature? To our knowledge, it is the only value close to the Shapley value that obeys component efficiency and takes outside options into account. Arguably, there are many economic and political situations where we need these properties. For example, the power of a government coalition rests with the parties involved (component efficiency) and the power of each party within the government depends on other governments that might possibly form (outside options).

Close to the AD-approach, our outside-option value obeys component efficiency, symmetry and additivity. However, we argue that values \( \varphi^{oo} \) modelling component efficiency and outside options cannot possibly obey the null-player axiom. Consider \( N = \{1, 2, 3\} \) and the unanimity game \( u_{\{1,2\}} \) which maps the worth 1 to coalitions \( \{1, 2\} \) and \( \{1, 2, 3\} \) and the worth 0 to all other coalitions. We now look at the coalition structure \( \mathcal{P}_1 = \{\{1, 3\}, \{2\}\} \). By component efficiency, we get \( \varphi_1^{oo}(u_{\{1,2\}}, \mathcal{P}_1) + \varphi_3^{oo}(u_{\{1,2\}}, \mathcal{P}_1) = 0 = \varphi_2^{oo}(u_{\{1,2\}}, \mathcal{P}_1) \). Player 3 is a null player; his contribution to any coalition is zero. Yet, his payoff cannot be zero under \( \varphi^{oo} \). The reason is this: Player 1 has outside options. By joining forces with player 2 (thus violating the existing coalition structure) he would have claim to a payoff of 1/2. Within the existing coalition structure, he will turn to player 3 to satisfy at least part of this claim. But then, player 3’s payoff is negative.

It should also be clear that a component-efficient value that respects outside options cannot always coincide with the value for some "stable" partition. In our example, stable partitions might be given by \( \mathcal{P}_2 = \{\{1, 2\}, \{3\}\} \) or \( \mathcal{P}_3 = \)
By component efficiency the sum of payoffs for all three players is zero for $P_1$ but 1 for $P_2$ and $P_3$.

Some readers might object to a negative payoff for player 3 by pointing to the possibility that player 3 departs from coalition $\{1,3\}$ to obtain the zero payoff. However, for the purpose of determining the outside-option value, the coalition structure $P$ is given. The stability of $P$ is another -separate- issue that we will deal with in section 5.3. We also note that every player will obtain a non-negative payoff under the value presented in this paper in case of three conditions. First, the coalition function is simple and monotonic. Second, one component contains the government coalition (i.e. a set of parties with worth 1). Third, the other parties form singleton components. In this sense, the issue of negative payments need not concern us here. This result, presented in section 5.2, and another non-negativity result from section 5.3 are certainly most welcome. After all, a power index allowing for negative payoffs would be difficult to defend.

Another objection points to the example of the game $(-1) u_{\{1,2\}}$ together with the above partition $P_1 = \{\{1,3\}, \{2\}\}$. Here, player 1’s outside options are negative. If he were to join player 2, he would receive $-1/2$ and within the existing partition a component efficient value with outside options should attribute a negative payoff to player 1. A somewhat satisfactory interpretation could go as follows: Player 3 argues that player 1’s payoff were negative if he would form a coalition with player 2. It is the existing component $\{1,3\}$ that prevents this negative payoff. Part of this gain should then go to player 3.

In our paper, the null-player axiom is substituted by the outside-option axiom. In the literature, different alternatives to the null-player axiom can be found. For example, Nowak & Radzik (1994) present a solidarity value where null players in unanimity games obtain a positive value. (Their value could easily be turned into a partitional one.) A very different approach is that by Napel & Widgren (2001). For the class of simple games they define so-called inferior players who form a superset of the set of null players. All inferior players get a payoff of zero according to their Strict Power Index, a close relative of the Banzhaf index.

While our paper is coached in terms of the number of senator posts, power within government coalition can take many forms. It has long been recognized that it is useful to distinguish office payoffs (which include cabinet portfolio payoffs and patronage) and policy payoffs (referring to the desire of political parties to influence policy). For example, attacking the office-payoff problem, a recent paper by Carmignani (2001) uses the war of attrition (non-cooperative game theory) to derive and test hypotheses about the duration of the cabinet-formation process,
the portfolios secured by the so-called formateur etc. On the other hand, Warwick (2001) contributes to the policy-payoff literature. It should be noted that our paper is not sold on either of the two approaches.

Another solution to the quantitative cabinet-portfolio problem is Brams & Kaplan (2002), who present and improve a procedure used in Northern Ireland that attributes positive integers to political parties. In fact, they argue that “because it is persons that become ministers, we cannot give fractional numbers of ministries to parties”. However, even with people, divisibility can be achieved, for example, by time sharing models as discussed for the presidency of the European Central Bank, by tossing a coin, or, as in our example, by letting several parties having a say in filling a post.

The paper is organized as follows: In section 2 basic definitions are given. Section 3 presents some ”usual” axioms for coalition-structure values while section 4 introduces an axiom that reflects outside options of players. The outside-option value is presented and axiomatized in section 5. Section 6 then presents the application of the outside-option value to the analysis of the elections in Berlin. Section 7 concludes the paper.

2. Definitions

Let \( N = \{1, 2, ..., n\} \) be the player set. A game (in characteristic function form) is a pair \( (N, v) \) where \( v \) is a function \( 2^N \rightarrow \mathbb{R} \) such that \( v(\emptyset) = 0 \). \( v \) is also called a coalition function. The set of all games on \( N \) is denoted \( G \).

A payoff vector \( x \) for \( N \) is an element of \( \mathbb{R}^n \) or a function \( N \rightarrow \mathbb{R} \). As usual, we abbreviate \( \sum_{j \in S} x_j \) by \( x(S) \) for all subsets \( S \) of \( N \) and let \( x(\emptyset) = 0 \).

A game \( v \) is called monotonic if for any two coalitions \( K, K' \) fulfilling \( \emptyset \subseteq K \subseteq K' \subseteq N \) we get \( v(K) \leq v(K') \). \( v \) is a simple game if \( v(K) \in \{0, 1\} \) for all \( K \subseteq N \). For simple games, any coalition \( K \) fulfilling \( v(K) = 1 \) is called a winning coalition. The set of winning coalitions for \( v \) is denoted by \( W(v) \). Player \( i_{\text{veto}} \) is called a veto player if \( i_{\text{veto}} \in W \) for all \( W \in W(v) \) holds. A simple monotonic game is called noncontradictory if \( W \in W(v) \) implies \( N \setminus W \notin W(v) \).

A game \( v \) is called convex if for any two coalitions \( K, K' \) fulfilling \( \emptyset \subseteq K \subseteq K' \subseteq N \) and for all players \( i \in N \) fulfilling \( i \notin K' \), we have

\[
v(K \cup \{i\}) - v(K) \leq v(K' \cup \{i\}) - v(K').
\]

Thus, the marginal contribution of player \( i \) is higher for \( K' \) than for \( K \subseteq K' \).
For any nonempty coalition $T \subseteq N$, $u_T (S) = 1$, $S \supseteq T$; 0 otherwise, defines a game, called a unanimity game. It is well known that the set of those games (the cardinality of which is $2^n - 1$) is a basis of $G$ in the sense of linear algebra.

A game $(N, v)$ is called symmetric, if a function $f : N \rightarrow \mathbb{R}$ exists such that $v (S) = f (|S|)$ for all nonempty sets $S \subseteq N$. A player $i \in N$ is a null player for $v \in G$ if $v (S \cup \{i\}) = v (S)$ for all $S \subseteq N$.

Following Aumann & Drèze (1974), we define coalition structures: A coalition structure $\mathcal{P}$ on $N$ (sometimes written as $(N, \mathcal{P})$) is a partition of $N$ into components $C_1, ..., C_m$:

$$\mathcal{P} = \{C_1, ..., C_m\}.$$  

Thus, $\bigcup_{j=1}^m C_j = N$, $C_j \cap C_k = \emptyset$ for all $j, k \in \{1, \ldots, m\}$, $j \neq k$ and $C_j \neq \emptyset$ for all $j \in \{1, \ldots, m\}$. The set of all partitions on $N$ is denoted by $\mathfrak{P}$. For any player $i \in N$, $\mathcal{P} (i)$ denotes the component containing $i$. For any set $T \subseteq N$, the set of components containing any players from $T$ is written $\mathcal{P} (T)$. These components are called $T$-components. (The reader will note $\mathcal{P} (i) = \mathcal{P} (j)$ and for all coalitions $K$ obeying $i \notin K$ and $j \notin K$ we have

$$v (K \cup \{i\}) = v (K \cup \{j\}).$$

Rank orders $\rho$ on $N$ are written as $(\rho_1, \ldots, \rho_n)$ where $\rho_1$ is the first player in the order, $\rho_2$ the second player etc. The set of all rank orders on $N$ is denoted by $R$. For every $i \in N$ there exists a $j (i) \in N$ such that $\rho_{j(i)} = i$. Then, we define $K_i (\rho) := \{\rho_1, \ldots, \rho_{j(i)}\}$. Consider, now, a coalition $L \subseteq N$ fulfilling $i \in L$ and $L \subseteq K_i (\rho)$. There exists a rank order $\rho^L$ such that $L = K_i (\rho^L)$ and such that player $\ell \in L$ precedes player $\ell' \in L$ in $\rho^L$ iff $\ell$ precedes $\ell'$ in $\rho$.

The Shapley value and other related values make heavy use of marginal contributions of players. For any coalition $S \subseteq N$ and any player $i \in S$ we define

$$MC_i^S (v) := v (S) - v (S \setminus \{i\})$$

and, given some rank order $\rho$ from $R$,

$$MC_i (v, \rho) := v (K_i (\rho)) - v (K_i (\rho) \setminus \{i\}).$$
3. Axioms

A value on \((N, \mathfrak{P})\) is a function \(\psi : G \times \mathfrak{P} \to \mathbb{R}^n\). Values on \((N, \mathfrak{P})\) might obey one or several of the following axioms:

**Axiom E (Efficiency):**

\[
\sum_{i \in N} \psi_i (v, \mathcal{P}) = \psi (v, \mathcal{P}) (N) = v (N)
\]

**Axiom CE (Component efficiency):** For all \(C \in \mathcal{P}\),

\[
\sum_{i \in C} \psi_i (v, \mathcal{P}) = \psi (v, \mathcal{P}) (C) = v (C).
\]

**Axiom S (Symmetry):** For all players \(i\) and \(j\), symmetric with respect to partition \(\mathcal{P}\),

\[
\psi_i (v, \mathcal{P}) = \psi_j (v, \mathcal{P}) .
\]

**Axiom N (Null player):** For any null player \(i \in N\),

\[
\psi_i (v, \mathcal{P}) = 0.
\]

**Axiom N-S (Null player):** For any nonempty set \(T \subseteq N\),

\[
\psi (u_T, \mathcal{P}) (T^c) = 0,
\]

where \(T^c := N \setminus T\).

**Axiom N-AD (Null player):** For any nonempty set \(T \subseteq N\) and any \(C \in \mathcal{P}\),

\[
\psi (u_T, \mathcal{P}) (C \cap T^c) = 0 .
\]

**Axiom A (Additivity):** For any coalition functions \(v_1, v_2 \in G\),

\[
\psi (v_1 + v_2, \mathcal{P}) = \psi (v_1, \mathcal{P}) + \psi (v_2, \mathcal{P})
\]

**Axiom L (Linearity):** For any coalition functions \(v_1, v_2 \in G\) and any \(\alpha \in \mathbb{R}\),

\[
\psi (\alpha v_1 + v_2, \mathcal{P}) = \alpha \psi (v_1, \mathcal{P}) + \psi (v_2, \mathcal{P})
\]

The reader will note that we make use of three null-player axioms. According to axiom N, any null player should receive a zero payoff. Axiom N-S claims that the sum of payoffs to the null players in a unanimity game should be zero. Finally,
axiom N-AD applies axiom N-S to all the components of a given partition \( \mathcal{P} \). Of course, axiom N implies axioms N-S and N-AD.

As is very well known (see Shapley (1953)), for \( \mathcal{P} := \{ N \} \), there exists a unique value satisfying the axioms E (or CE), S, N (or N-S), and A (or L), the Shapley value, written \( \varphi (v) \) for \( v \in G \). It is given by

\[
\varphi_i (v) = \frac{1}{n!} \sum_{\rho \in \mathcal{R}} MC_i (v, \rho), i \in N.
\]

The AD-value is the Shapley value gained by restricting the coalition function to the components of a partition \( \mathcal{P} \):

\[
\varphi^{AD} (v, \mathcal{P}) := (\varphi_i (v|_{\mathcal{P}(i)}))_{i \in N}.
\]

The AD-value (see Aumann & Drèze (1974)) is uniquely determined by the axioms CE, S, N (or N-AD), and A (or L).

4. An outside-option axiom for unanimity games

In this section, we will present an axiom needed to axiomatize the outside-option value. Let \((N, \mathcal{P})\) be a coalition structure and \( T \subseteq N \) a nonempty set.

**Axiom N-oo (Outside options for unanimity games):** If \( \mathcal{P} \) contains a component \( C_T \) such that \( T \subseteq C_T \), then for all \( C \in \mathcal{P} \),

\[
\psi (u_T, \mathcal{P}) (C \cap T^c) = 0.
\]

If \( \mathcal{P} \) does not contain a component \( C \) such that \( T \subseteq C \), then for all \( C \in \mathcal{P} \),

\[
\psi (u_T, \mathcal{P}) (C \cap T^c) = -\frac{|C \cap T| |C \cap T^c|}{|T| |T \cup C|}.
\]

This axiom corresponds to the null-player axiom N. In fact, if there is a component \( C_T \) of \( \mathcal{P} \) that contains \( T \), all (symmetric!) null players receive a payoff of 0 in \( u_T \). This holds for all the players in \( N \setminus C_T \) and for those in \( C_T \setminus T \). If, however, such a component does not exist, the players from \( T \) find themselves in two or more components of \( \mathcal{P} \). Then the worth of each component is zero. However, players from \( T \) might arguably not be content with a payoff of zero. If
they were not bound to the component they happen to find themselves in, they might possibly take part in a coalition promising to divide the worth of 1. By component efficiency, for every component $C$ not containing $T$, positive payoffs for players from $T \cap C$ necessitate negative payoffs for players from $T^c \cap C$. Of course, players from $T^c$ with negative payoffs could possibly better their lot by leaving their component. We will address the stability problem in the next section.

Note that payoffs to players in $C \cap T$ are zero if $C \cap T = \emptyset$ because in this case there are no $T$-players in $C$ threatening to look for other coalitions.

Axiom N-oo makes use of $\frac{|C \cap T|}{|T|}$; this term can be interpreted as the probability that a player from $C \cap T$ (as opposed to a player from $C \cap T^c$) claims the unit payoff. The second factor reflects the probability that players from $C \cap T$ actually do have to make up for the missed opportunities of $T$-players belonging to the same component. The denominator $|T \cup C|$ of the second factor is somewhat unsatisfactory. Instead, one might prefer an alternative axiom which is like N-oo but attributes $\psi(u_T, \mathcal{P})(C \cap T^c) = -\frac{|C \cap T^c|}{|C|}$ if no component $C$ of $\mathcal{P}$ can be found that fulfills $T \subseteq C$. $\frac{|C \cap T^c|}{|C|}$ has the simple interpretation of letting all the players from $C$ (rather than all the players from $C \cup T$ as in N-oo) have equal probability for paying a $T$-player who happens to complete $T$ and to gain the unit payoff. Alas, an axiomatization along the lines presented in the next section could not be found.

Note that axioms CE, S, and N-oo imply

$$
\psi_i(u_T, \mathcal{P}) = \begin{cases} 
\frac{1}{|P(i)|}, & i \in T \text{ and } \exists C \in \mathcal{P} : T \subseteq C, \\
0, & i \notin T \text{ and } \exists C \in \mathcal{P} : T \subseteq C, \\
\frac{1}{|P(i)|} \frac{|P(i) \cap T^c|}{|P(i)|}, & i \in T \text{ and } \exists C \in \mathcal{P} : T \subseteq C, \\
-\frac{1}{|P(i)|} \frac{|P(i) \cap T^c|}{|P(i)|}, & i \notin T \text{ and } \exists C \in \mathcal{P} : T \subseteq C.
\end{cases}
$$

(4.1)

5. The outside-option value

5.1. The formula and its axiomatization

We will now define the outside-option value $\varphi_i^{oo}$. It is given by

$$
\varphi_i^{oo}(v, \mathcal{P}) = \frac{1}{n!} \sum_{\rho \in \mathcal{R}} \left( v(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i) \setminus \{i\}} MC_j(v, \rho), \quad \mathcal{P}(i) \subseteq K_i(\rho), \quad \text{otherwise},
\right)
$$

(5.1)
The reader notes that player i’s payoff does not depend on the partition \( \mathcal{P} \) in general, but only on \( \mathcal{P}(i) \). In looking at a rank order \( \rho \), player \( i \) gets her marginal contribution \( MC_i(v, \rho) \) if she is not the last player in her component in \( \rho \), i.e., if \( \mathcal{P}(i) \) is not included in \( K_i(\rho) \). If \( i \) is the last player in her component, she gets the worth of this component minus the payoffs (marginal contributions \( MC_j(v, \rho) \)) to the other players in her component.

The above formula lends itself to an interpretation very close to the one given for the Shapley value. For both formulae, we consider that all players arrive in a random order. For the Shapley value, the player’s receive their marginal contribution with respect to the players arriving before them. In our formula, matters are a bit more complicated. For every rank order \( \rho \), exactly one player \( i \) from \( \mathcal{P}(i) \) is not followed by other players from her component. The other players from \( \mathcal{P}(i) \setminus \{i\} \) get their marginal contributions as in the Shapley case. This marginal contribution will not always concern players from \( \mathcal{P}(i) \) exclusively. Some of the players in \( K_j(\rho), j \in \mathcal{P}(i) \setminus \{i\} \), may well be outside \( \mathcal{P}(i) = \mathcal{P}(j) \) so that outside options are taken into account. Player \( i \), who is the last player in her component, obtains the worth of her component net of the marginal contributions awarded to the other players in her component.

**Theorem 5.1.** \( \phi^{oo} \) is the unique value on \((N, \mathfrak{P})\) satisfying the axioms CE, S, N-oo, and L. \( \blacksquare \)

The proof is given in the appendix.

### 5.2. The outside-option value and the Shapley value

The outside-option value is a generalization of the Shapley value in two senses. First, \( \mathcal{P} = \{N\} \) yields the Shapley value for any coalition function, \( \phi^{oo}_i(v, \{N\}) = \phi_i(v) \). Second, a (big) party that necessarily belongs to every government coalition obtains the Shapley value, as the first part of the following proposition shows. By pointing to its outside options, this party is indifferent between all winning coalitions.

The second part of the proposition makes clear that political parties cannot normally obtain a negative power under \( \phi^{oo} \). In view of the negative payoffs in axiom N-oo, this is certainly a welcome result for a power index.

**Proposition 5.2.** Let \((v, N)\) be a simple and monotonic game and \( \mathbb{W}(v) \) its set of winning coalitions.
1. Let there be a veto player \( i_{\text{veto}} \in N \), i.e. \( i_{\text{veto}} \in W \) for all \( W \in \mathcal{W} (v) \). Let \( \mathcal{P} \) be a partition of \( N \) such that \( \mathcal{P}(i_{\text{veto}}) \in \mathcal{W}(v) \). Then, \( \varphi_{i_{\text{veto}}}^\infty (v, \mathcal{P}) = \varphi_i (v) \).

2. Let \( \mathcal{P} \) be a partition such that \( \mathcal{P} \) contains exactly one winning coalition \( W \) (the government coalition) and such that \( |\mathcal{P}(j)| = 1 \) for all players \( j \notin W \). Then \( \varphi_i^\infty (v, \mathcal{P}) \geq 0 \) for all players \( i \in N \). □

The proof is relegated to the appendix.

5.3. Attacking the stability problem

The question of stable partition structures is surely of utmost importance. Adapting the definition proposed by Hart & Kurz (1983), we define stability in the following manner:

**Definition 5.3.** A coalition structure \( \mathcal{P} \) is stable for \( \varphi^\infty \) if there is no coalition \( L \) such that all players from \( L \) profit from forming a component, i.e. if for all \( L \neq \emptyset \) we have

\[
\varphi_i^\infty (v, \mathcal{P}) \geq \varphi_i^\infty (v, \{L, N\setminus L\}) \text{ for some } i \in L,
\]

or, equivalently, if for all \( L \neq \emptyset \) we have

\[
\varphi_i^\infty (v, \mathcal{P}) \geq \varphi_i^\infty (v, \mathcal{P}^L) \text{ for some } i \in L.
\]

Our definition is simpler than the original one. The reason is that the outside option value for a player in component \( C \) is not influenced by how the players outside \( C \) are partitioned.

We will present two sets of stability results. The first one refers to games in general, the second to weighted majority games.

**Proposition 5.4.** Stable coalition structures for \( \varphi^\infty \) exist for all symmetric and for all convex games. Also, a stable partition \( \mathcal{P} \) fulfills \( \varphi_i^\infty (v, \mathcal{P}) \geq v (\{i\}) \) for all players \( i \in N \).

The proof can be found in the appendix.

We now turn to weighted majority games. For \( n \) players, these are given by

\[
[g_1, \ldots, g_n]
\]
where $g_1 \geq 0$ through $g_n \geq 0$ are the weights of players 1 through $n$ with $g_i > 0$ for at least one player $i \in \{1, \ldots, n\}$. The corresponding coalition function is given

$$v(K) = \begin{cases} 1, & \sum_{i \in K} g_i > \frac{\sum_{i \in N \setminus K} g_i}{2} \\ 0, & \sum_{i \in K} g_i \leq \frac{\sum_{i \in N \setminus K} g_i}{2} \end{cases}$$

Weighted majority games are noncontradictory monotonic simple games. Numerous examples with up to eight players might indicate that weighted majority games always possess stable partition structures.

**Conjecture 5.5.** All weighted majority games possess a stable partition structure.

It is not yet clear whether this conjecture holds. We will report on some progress towards proving it. The following lemma shows that instability of a partition can be shown (if at all) via minimal winning coalitions:

**Lemma 5.6.** Let $W \in \mathbb{W}(v)$ be a winning coalition of a weighted majority game $v$ and $i \in W$. Then, for all players $k \neq i$,

$$\varphi^\infty_i (v, \{W, N \setminus W\}) \geq \varphi^\infty_i (v, \{W \cup \{k\}, N \setminus (W \cup \{k\})) \}.$$

Again, turn to the appendix for a proof.

In view of the above lemma it is tempting to look for stable partitions of weighted majority games by considering minimal winning coalitions. However, minimal winning coalitions need not be stable if we limit attention to low-weight players or if we limit attention to high-weight players.

Consider $n = 5$ and the weighted majority game $[5, 4, 3, 2, 1]$. Coalitions $\{2, 3, 4, 5\}$ and $\{1, 2\}$ are minimal winning coalitions with payoffs

- \[
\begin{pmatrix}
0, \frac{7}{20}, \frac{7}{20}, \frac{3}{20}, \frac{3}{20}
\end{pmatrix}
\] for $\mathcal{P}^{(2,3,4,5)}$
- \[
\begin{pmatrix}
\frac{7}{12}, \frac{5}{12}, 0, 0, 0
\end{pmatrix}
\] for $\mathcal{P}^{(1,2)}$

Since $\frac{5}{12} > \frac{7}{20}$, $\mathcal{P}^{(2,3,4,5)}$ is not a stable partition.
Now consider the weighted majority game \([7, 5, 4, 4, 4]\). Coalitions \(\{1, 2, 3\}\) and \(\{1, 4, 5\}\) are minimal winning coalitions with payoffs

\[
\begin{pmatrix}
  \frac{7}{20} & \frac{7}{20} & \frac{3}{10} & 0 & 0 \\
  \frac{2}{5} & 0 & 0 & \frac{3}{10} & \frac{3}{10}
\end{pmatrix}
\]

for \(P^{\{1,2,3\}}\) and

\[
\begin{pmatrix}
  \frac{7}{20} & \frac{7}{20} & \frac{3}{10} & 0 & 0 \\
  \frac{2}{5} & 0 & 0 & \frac{3}{10} & \frac{3}{10}
\end{pmatrix}
\]

for \(P^{\{1,4,5\}}\).

Thus, \(P^{\{1,2,3\}}\) is not a stable partition.

We close this section with some positive stability results:

**Proposition 5.7.** Let \(v\) be the coalition function for the weighted majority game \([g_1, \ldots, g_n]\).  

- If \([g_1, \ldots, g_n]\) gives rise to a symmetric coalition function \(v\), stable partitions exist.

- Player \(i_{\text{veto}}\) is a veto player in \(v\) iff \(g_{i_{\text{veto}}} \geq \sum_{i \in N, i \neq i_{\text{veto}}} g_i\). Any partition \(P\) fulfilling \(P(i_{\text{veto}}) \in W(v)\) for a veto player \(i_{\text{veto}}\) is stable.

- All weighted majority games with five or less players possess stable partitions.

See the appendix for a proof.

Let us consider the weighted majority game \([3, 1, 1, 1, 1]\), analyzed by Aumann & Myerson (1988). Applying a different concept (sequential link formation game and Myerson value), they argue for the winning coalition \(\{2, 3, 4, 5\}\) where the small players join to beat the big one. The outside-option value yields the payoffs

\[
\begin{pmatrix}
  \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\
  \frac{7}{10} & \frac{3}{20} & \frac{3}{20} & 0 & 0 \\
  \frac{13}{20} & \frac{7}{60} & \frac{7}{60} & \frac{7}{60} & 0 \\
  \frac{3}{5} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\
  0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}
\]

for \(P^{\{1,2\}}\), \(P^{\{1,2,3\}}\), \(P^{\{1,2,3,4\}}\), \(P^{\{1,2,3,4,5\}}\), and \(P^{\{2,3,4,5\}}\), respectively.
where $\frac{3}{4} > \frac{7}{10} > \frac{13}{20} > \frac{3}{5}$ and $\frac{1}{4} > \frac{3}{20} > \frac{7}{60} > \frac{1}{10}$ hold in accordance with lemma 5.6. Therefore, $\{\mathcal{P}^{\{1,j\}} : j = 2, \ldots, 5\} \cup \{\mathcal{P}^{\{2,3,4,5\}}\}$ is the set of stable partitions.

6. The elections in Berlin 2001

For illustrative purposes, we comment on the elections for the Berlin house of representatives that took place in 2001. Following these elections, five parties were present in the house, the social democrates (SPD), the Christian democrates (CDU), the socialists (PDS), the liberals (FDP) and the green party (Bündnis 90/Die Grünen). The number of seats and the percentage points are given in the following table:

<table>
<thead>
<tr>
<th>Party</th>
<th>Number of Seats</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPD</td>
<td>44</td>
<td>31</td>
</tr>
<tr>
<td>CDU</td>
<td>35</td>
<td>25</td>
</tr>
<tr>
<td>PDS</td>
<td>33</td>
<td>23</td>
</tr>
<tr>
<td>FDP</td>
<td>15</td>
<td>11</td>
</tr>
<tr>
<td>Green</td>
<td>14</td>
<td>10</td>
</tr>
</tbody>
</table>

In general, a government coalition needs at least 50% of the seats. However, for general political reasons neither the CDU nor the FDP would want to be involved in a government coalition that contains the PDS. Also, the SPD had promised its voters not to form a coalition with the CDU. The ”grand coalition” SPD/CDU had existed for some years and the Berlin electorate did not want to see these parties ruling together again. For these reasons, the simple game describing the 2001 election is given by

$$v(K) = \begin{cases} 
1, & K \supseteq \{\text{SPD, PDS}\}, \\
1, & K \supseteq \{\text{SPD, FDP, Green}\}, \\
0, & \text{otherwise.}
\end{cases}$$

At first, the SPD had coalition talks with the liberal party and the green party. These talks failed. SPD and PDS then formed a government coalition giving the mayor’s office and 5 senator posts to the SPD and 3 senator posts to the PDS.

One cannot expect the Shapley value to reflect the outcome of these talks because this value measures power ex ante. All parties necessary to at least one
government coalition obtain a positive value:

<table>
<thead>
<tr>
<th></th>
<th>SPD 7/12</th>
<th>CDU 0</th>
<th>PDS 1/4</th>
<th>FDP 1/12</th>
<th>Green 1/12</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPD</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>CDU</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>PDS</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>FDP</td>
<td>0</td>
<td>0</td>
<td>5/24</td>
<td>0</td>
<td>5/24</td>
</tr>
<tr>
<td>Green</td>
<td>0</td>
<td>0 (!)</td>
<td>1/12</td>
<td>5/24</td>
<td>0</td>
</tr>
</tbody>
</table>

During the course of these negotiations, the SPD hinted at the possibility of letting the Greens into the government as well, i.e. to form a government SPD-PDS-Green. Therefore, we will consider three alternative government coalitions.

The following table informs about the AD-value associated with these three government coalitions.

<table>
<thead>
<tr>
<th></th>
<th>SPD-PDS 7/12</th>
<th>SPD-PDS-Green 1/2</th>
<th>SPD-FDP-Green 1/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPD</td>
<td>1/2</td>
<td>1/2</td>
<td>1/3</td>
</tr>
<tr>
<td>CDU</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>PDS</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>FDP</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
</tr>
<tr>
<td>Green</td>
<td>0</td>
<td>0 (!)</td>
<td>1/3</td>
</tr>
</tbody>
</table>

According to the AD-value only parties that belong to a government coalition can have a positive value, i.e. the AD-value takes the ex-post perspective. Interestingly, the power of the Greens within a government coalition of SPD and PDS is zero; the Greens are not necessary to the SPD-PDS government. Note that the AD-value is not sensitive to outside options.

The outside-option value generates the following results:

<table>
<thead>
<tr>
<th></th>
<th>SPD-PDS 7/12</th>
<th>SPD-PDS-Green 7/12</th>
<th>SPD-FDP-Green 7/12</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPD</td>
<td>7/12</td>
<td>7/12</td>
<td>7/12</td>
</tr>
<tr>
<td>CDU</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>PDS</td>
<td>5/12</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>FDP</td>
<td>0</td>
<td>0</td>
<td>5/24</td>
</tr>
<tr>
<td>Green</td>
<td>0</td>
<td>1/12</td>
<td>5/24</td>
</tr>
</tbody>
</table>

Five aspects are noteworthy:

1. The SPD obtains its Shapley value within every government coalition as noted in proposition 5.2.
2. The government coalition SPD-PDS-Green is worse for the PDS than the government coalition SPD-PDS. In fact, the PDS did not like the idea to admit the Greens to the government as well.

3. By the outside options, within the SPD-PDS-Green government the Greens would not be powerless.

4. Comparing the two right-hand columns, the alternative government SPD-PDS-Green does not give more power to the Greens than to the FDP within the SPD-FDP-Green government. The SPD-PDS-Green government is not an outside option for the Greens who engage in the SPD-FDP-Green government.

5. According to the outside-option value, Mr. Gysi should have expected to fill 3.75 senator posts. Therefore, he could not be content with only 3 senator posts. As mentioned in the opening paragraph, Mr. Gysi obtained nearly half another senator post which brought him to 3.5 senator posts.

7. Conclusions

In this paper, we develop an ex-post power index that distributes the power of 1 to the parties forming a government coalition. The main idea of our index is this: The power rests with the parties involved (component efficiency) but the power of each party within the government depends on other governments that might possibly form (outside options).

With respect to the 2001 elections in Berlin, the outside-option value performs quite satisfactorily. Of course, some empirical work (along the lines described in chapter 7 by Laver & Schofield (1990)) would be needed to gain more confidence in this new value.

Apart from political theory, the outside-option value has other interesting applications. For example, in markets one might be interested in an ex-post value that would give us an idea about the payoff to market participants once they have or have not found a trading partner. In particular, the outside-option value could be used to make predictions about the price.

8. Appendix

Proof of theorem 5.1:
As usual, the proof has to state two things. First, the value fulfills the axioms; second, there is no other value to do so. The second part is standard and makes heavy use of linearity and the fact that the unanimity games form a basis of \( G \), e.g. Aumann (1989, pp. 30).

We will contend ourselves to comment on the interesting aspects of the first part. Axioms CE, S, and L are easily checked as is the first part of axiom N-oo. Turning to the second part of axiom N-oo, we fix a partition \( \mathcal{P} = \{C_1, ..., C_m\} \) and a nonempty set \( T \subseteq N \) and assume that \( T \) is not contained in any of \( \mathcal{P} \)'s components. Now, for a given \( \rho \in R \) and a given component \( C \) from \( \mathcal{P} \), if a player \( i \) from \( C \cap T^c \) is not the last of his component to occur in \( \rho \) (i.e. \( \mathcal{P} (i) \not\subseteq K_i (\rho) \)), he receives zero. If our player from \( C \cap T^c \) happens to be the last \( C \)-player and to occur after all players from \( T \), he has to pay \(-1\) if, with probability \( \frac{|C|}{|T|} \), a player from \( C \cap T \) was the last of all the \( T \)-players to occur in \( \rho \). Therefore, players from \( C \cap T^c \) obtain \(-\frac{|C|}{|T|} \).

**Proof of proposition 5.2:**

As to the first part of the proposition, we can show that the Shapley formula and our formula attribute the same payoff to player \( i_{\text{veto}} \) for each rank order \( \rho \). Let \( \rho \) be a rank order fulfilling \( \mathcal{P} (i_{\text{veto}}) \subseteq K_{i_{\text{veto}}} (\rho) \). Then, \( v (\mathcal{P} (i_{\text{veto}})) = 1 \) and \( MC_j (v, \rho) = 0 \) for all players from \( j \in \mathcal{P} (i_{\text{veto}}) \setminus \{i_{\text{veto}}\} \) in our formula and \( MC_{i_{\text{veto}}} (v, \rho) = 1 \) in the Shapley formula. If \( \rho \) does not fulfill \( \mathcal{P} (i_{\text{veto}}) \subseteq K_{i_{\text{veto}}} (\rho) \), both formulae attribute \( MC_{i_{\text{veto}}} (v, \rho) \) to player \( i_{\text{veto}} \).

We now turn to the second part of the proposition. By component efficiency, we have \( \varphi_{j}^{oo} (v, \mathcal{P}) = v (\mathcal{P} (j)) = 0 \) for all players \( j \not\in W \). Let \( i \) be any player from \( W \). If \( \rho \) is a rank order fulfilling \( W = \mathcal{P} (i) \subseteq K_i (\rho) \), we have \( v (\mathcal{P} (i)) = 1 \) and \( MC_j (v, \rho) = 1 \) for at most one player \( j \) from \( \mathcal{P} (i) \setminus \{i\} \). For such a \( \rho \), \( i \)'s payoff is 0 or 1. If \( \rho \) does not fulfill \( W = \mathcal{P} (i) \subseteq K_i (\rho) \), \( i \) gets his marginal contribution \( MC_i (v, \rho) \) which is also equal to 0 or 1. Hence, \( \varphi_{i}^{oo} (v, \mathcal{P}) \geq 0 \) for players \( i \in W \).

**Proof of proposition 5.4:**

Let \( v \) be a symmetric game, \( i' \in N \) any player and \( \mathcal{P}' \) any partition from \( \arg \max_{\mathcal{P} \in \mathcal{P}} \varphi_{i'}^{oo} (v, \mathcal{P}) \). No player from \( \mathcal{P}' (i') \) can obtain more than \( \varphi_{i'}^{oo} (v, \mathcal{P}') \) in any component of any partition of \( N \). Let \( i'' \in N \setminus \mathcal{P}' (i') \) be any player and \( \mathcal{P}'' \) be any partition from \( \arg \max_{\mathcal{P} \in \mathcal{P}} \varphi_{i''}^{oo} (v, \mathcal{P}) \). Given that players from \( \mathcal{P}' (i') \) are content to form a component for themselves, no player from \( \mathcal{P}'' (i'') \) can obtain more than \( \varphi_{i''}^{oo} (v, \mathcal{P}'') \) in any component of any partition of \( N \setminus \mathcal{P}' (i') \). Pursuing in this manner, we arrive at a partition \( \{\mathcal{P}' (i'), \mathcal{P}'' (i''), ...\} \).
that is stable.

Now, let \( v \) be a convex game. We will show that \( \mathcal{P} = \{N\} \) is a stable partition. Consider any coalition \( L \neq \emptyset \) and the partition \( \{L, N \setminus L\} \). We need to show that there is some \( i \in L \) such that \( \varphi_i^o (v, \{N\}) \geq \varphi_i^o (v, \{L, N \setminus L\}) \). Indeed, we can show this inequality for any \( i \in L \). Thus, consider any \( i \in L \) and any rank order \( \rho \) fulfilling \( L \subseteq K_i (\rho) \) so that player \( i \) is the last player in \( L \) in \( \rho \). By convexity of \( v \), we have
\[
v(L) = \sum_{j \in L} MC_j (v, \rho^L) \leq \sum_{j \in L} MC_j (v, \rho)
\]
and hence
\[
v(L) - \sum_{j \in L \setminus \{i\}} MC_j (v, \rho) \leq MC_i (v, \rho) .
\]
Referring back to the outside-option value, eq. 5.1, there is no rank order \( \rho \) such that player \( i \) gets more under \( \{L, N \setminus L\} \) than her marginal contribution. Therefore, \( \varphi_i^o (v, \{L, N \setminus L\}) \leq \varphi_i^o (v, \{N\}) \). This shows that convex games possess the stable partition \( \{N\} \).

Finally, the last result is an immediate consequence of component efficiency for one-player components.

**Proof of lemma 5.6:**

We distinguish three cases. First, consider rank orders \( \rho \) that do not fulfill \( W \subseteq K_i (\rho) \). Then, \( \rho \) does not fulfill \( W \cup \{k\} \subseteq K_i (\rho) \) and player \( i \) obtains \( MC_i (v, \rho) \) for both partitions.

Second, assume rank orders \( \rho \) fulfilling \( W \subseteq K_i (\rho) \), but not \( W \cup \{k\} \subseteq K_i (\rho) \). Player \( i \) obtains \( v(W) - \sum_{j \in W \setminus \{i\}} MC_j (v, \rho) \) under partition \( \{W, N \setminus W\} \) and \( MC_i (v, \rho) \) under partition \( \{W \cup \{k\}, N \setminus (W \cup \{k\})\} \). Consider two subcases.

(i) There exists a player \( j \in W \setminus \{i\} \) such that \( MC_j (v, \rho) = 1 \). Then, we get
\[
v(W) - \sum_{j \in W \setminus \{i\}} MC_j (v, \rho) = 1 - 1 = 0 = MC_i (v, \rho) .
\]

(ii) There does not exist any player \( j \in W \setminus \{i\} \) such that \( MC_j (v, \rho) = 1 \). We obtain
\[
v(W) - \sum_{j \in W \setminus \{i\}} MC_j (v, \rho) = 1 - 0 = 1 \geq MC_i (v, \rho) .
\]
In both subcases, player \( i \) obtains at least as much under \( \{W, N \setminus W\} \) than under \( \{W \cup \{k\}, N \setminus (W \cup \{k\})\} \).
Third, consider rank orders \( \rho \) fulfilling both \( W \subseteq K_i (\rho) \) and \( W \cup \{ k \} \subseteq K_i (\rho) \). The payoffs are

\[
v (W) - \sum_{j \in W \setminus \{ i \}} MC_j (v, \rho) = 1 - \sum_{j \in W \setminus \{ i \}} MC_j (v, \rho) \geq 1 - \sum_{j \in (W \cup \{ k \}) \setminus \{ i \}} MC_j (v, \rho) = v (W \cup \{ k \}) - \sum_{j \in (W \cup \{ k \}) \setminus \{ i \}} MC_j (v, \rho).
\]

Thus, the three cases prove that no player in a winning coalition can benefit if this coalition admits additional members.

**Proof of proposition 5.7:**

The first claim is a direct application of proposition 5.4 to weighted majority games. The first part of the second assertion is obvious. The stability of \( P \) fulfilling \( P (i_{\text{veto}}) \in W (v) \) goes back to proposition 5.2: \( i_{\text{veto}} \) obtains the Shapley value in any winning coalition in which he partakes.

We now turn to the third claim. The cases of 1, 2, or 3 players are dealt with easily. For four players, assume without loss of generality \( g_1 \geq g_2 \geq g_3 \geq g_4 \geq 0 \).

If \( g_1 = g_4 \) holds, stability follows from the first claim of this proposition. If \( g_1 > g_2 + g_3 + g_4 \) holds, player \( i \) is a veto player and the second assertion applies. Otherwise, \( \{ 1, 2 \} \) is a minimal winning coalition and \( P^{\{1,2\}} \) is a stable partition. This can be checked by comparing payoffs at \( P^{\{1,2\}} \) with payoffs at \( P^{\{1,3\}}, P^{\{1,4\}}, P^{\{1,3,4\}}, P^{\{2,3\}}, \) and \( P^{\{2,3,4\}} \). Note that \( \{ 2, 4 \} \) is not a winning coalition. Carrying out the necessary comparisons is tedious. However, by lemma 5.6 we are justified to assume that the coalitions \( \{ 1, 3 \} \) through \( \{ 2, 3, 4 \} \) are minimal winning coalitions.

The proof for 5 players is even more tedious. We present the basic steps:

- If \( g_1 = g_5 \) holds, stability follows as in the four-players case.
- If player 1 is a veto player, stability is again immediate from the second assertion.
- If neither of the two conditions above hold, \( \{ 1, 2 \} \) is a minimal winning coalition or \( \{ 1, 2, 3 \} \) is a minimal winning coalition.
• If \( \{1, 2\} \) is a minimal winning coalition, \( \mathcal{P}^{\{1,2\}} \) can be shown to be stable. This involves comparisons of payoffs for \( \mathcal{P}^{\{1,2\}} \) with payoffs at minimal (!) winning coalitions \( \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \) and \( \{2, 4, 5\} \). Coalitions \( \{2, 4\} \) and \( \{2, 5\} \) are not winning.

• If \( \{1, 2, 3\} \) is a minimal winning coalition, \( \mathcal{P}^{\{1,2,3\}} \) need not be stable as shown in section 5.3. Assume that \( \{2, 3, 5\} \) is a winning coalition. Then, it is a minimal winning coalition and \( \mathcal{P}^{\{2,3,5\}} \) is stable as can be shown by appropriate comparisons. If \( \{2, 3, 5\} \) is not a winning coalition, we have

\[
g_1 + g_4 \geq g_2 + g_3 + g_5 \ (\{2, 3, 5\} \text{ not winning})
\geq g_3 + g_4 + g_5 \ (g_2 \geq g_4)
\geq g_1 + g_2 \ (\{1, 2, 3\} \text{ minimal winning coalition})
\]

and hence \( g_2 = g_3 = g_4 \). Since \( \{1, 2\} \) is not a winning coalition, neither is \( \{1, 4\} \) and hence \( g_1 + g_4 = g_2 + g_3 + g_5 \). Therefore, \( g_1 = g_2 + g_5 \).

Now, in this case, assume \( \{2, 3, 4\} \) to be a winning coalition. Given the information we have gathered, \( \mathcal{P}^{\{2,3,4\}} \) can be shown to be stable. If, however, \( \{2, 3, 4\} \) is not a winning coalition, we conclude

\[
g_2 + g_3 + g_4 \leq g_1 + g_5 = (g_2 + g_5) + g_5
\]

and hence \( g_2 = g_5 \).

Therefore, the weighted majority game at hand is

\[
\begin{bmatrix}
g_1, \frac{g_1}{2}, \frac{g_1}{2}, \frac{g_1}{2}, \frac{g_1}{2}
g_1, \frac{g_1}{2}, \frac{g_1}{2}, \frac{g_1}{2}, \frac{g_1}{2}
\end{bmatrix}.
\]

For this game, \( \mathcal{P}^{\{1,2,3\}} \) is stable because of the payoffs

\[
\begin{pmatrix}
\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0, 0
\end{pmatrix}
\]

for \( \mathcal{P}^{\{1,2,3\}} \) and

\[
\begin{pmatrix}
0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}
\end{pmatrix}
\]

for \( \mathcal{P}^{\{2,3,4,5\}} \).
References


