UNIT-CONSISTENT AGGREGATIVE MULTIDIMENSIONAL INEQUALITY MEASURES: A CHARACTERIZATION

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Unit-Consistent Multidimensional Inequality
ABSTRACT

Inequality among people involves comparisons of social indicators such as income, health, education and so on. In recent years the number of studies both theoretical and empirical which take into account not only the individual’s income but also these other attributes has significantly increased. As a consequence the development of measures capable of capturing multidimensional inequality and satisfying reasonable axioms becomes a useful and important exercise.

The aim of this paper is no other than this. More precisely, we consider the unit consistency axiom proposed by B. Zheng in the unidimensional framework. This axiom demands that the inequality rankings, rather than the inequality cardinal values as the traditional scale invariance principle requires, are not altered when income is measured in different monetary units. We propose a natural generalization of this axiom in the multidimensional setting and characterize the class of aggregative multidimensional inequality measures which are unit-consistent.

Key Words: multidimensional inequality indices, unit-consistency, aggregativity.

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1. **INTRODUCTION**

This work takes as a reference two recent papers concerning inequality measurement. The first one is Zheng [20] who introduces a new unit consistency axiom in the unidimensional context and characterises families of inequality measures that fulfil this axiom. The other starting point is Tsui [17] who derives the class of multidimensional relative inequality measures.

There are several answers to the question of how to distribute an additional amount of income among the whole population without changing the initial inequality level. Whereas the rightist view, according to Kolm’s designation [9], demands a proportional distribution and asks that the inequality measure be scale invariant, the leftist view requires that inequality remains unchanged when each individual in the population receives the same amount of the extra income, and as a consequence, they insist that the inequality measure should fulfil the translation invariance principle. The centralist view, in turn, argues for a combination of these two answers. Examples of measures which correspond to this point of view can be found, among others, in Kolm [9], Bossert and Pfingsten [5], Seidl and Pfingsten [13], Chakravarty and Tyagarupananda ([6], [7]) and del Río and Ruiz-Castillo [8]. Actually, there is a justification behind the scale invariance principle: it makes no sense that inequality comparisons vary when income is measured in different monetary units.

Zheng [20] taking into account that none of these different points of view justifies an axiom for measuring income inequality with the obvious justification behind the scale invariance principle, introduces the unit consistency axiom in measuring inequality. This axiom requires that inequality orderings of income distributions do not change when incomes are measured in different monetary units. This appealing axiom is an ordinal property which allows inequality values to vary when monetary units change, provided the inequality orderings are not altered. Zheng characterises the class of both decomposable (Zheng [20]) and aggregative (Zheng [19]) unit-consistent inequality measures.

On the other hand several researchers (Kolm [10], Atkinson and Bourguignon [2], Tsui ([16], [17]), List [11], Savaglio [12], Weymark [18], among others) insist that in order to better answer the two questions posed by Sen [14]: “What is inequality?” and
“Inequality of what?” it is necessary to take into account several attributes in which people differ, such as income, health, education, and so on. Consequently, there is a need to extend axioms regarded as suitable in measuring income inequality to the multidimensional context and develop multidimensional inequality measures which are able to summarize inequalities as regards different attributes. Tsui [17] is a prominent example in this field. He proposes a correlation-increasing majorization criterion and derives the class of multidimensional generalized entropy measures.

In this paper we propose a straightforward extension of the unit-consistency axiom to the multidimensional setting and characterise the class of aggregative multidimensional inequality measures which are unit-consistent. The derived family is actually a generalization of the family characterised by Tsui [17].

The paper is structured as follows. The section below presents the notation and the definitions used in the paper. In Section 3 we introduce the generalization of the unit-consistency axiom to the multidimensional framework and present our characterisation results which are proved in the Appendix. Finally, Section 4 offers some concluding remarks. Most of the proofs of our paper follow both Zheng ([19], [20]) and Tsui [17] papers and the relevant results by Shorrocks [15] as well.

2. NOTATION AND BASIC AXIOMS OF MULTIDIMENSIONAL INEQUALITY INDICES

We consider a population consisting of \( n \geq 2 \) individuals endowed with \( k \geq 1 \) attributes. Let \( N = \{1, \ldots, n\} \) be the set of individuals and \( K = \{1, \ldots, k\} \) the set of attributes. A multidimensional distribution is represented by a \( n \times k \)-matrix \( X = (x_{ij}) \in M_+(n,k) \) over the non-negative real elements such that the sum of each column is non-zero, where \( x_{ij} \) represents individual \( i \)'s share of attribute \( j \). Then the \( ith \) row of \( X \), denoted by \( \vec{x}_i \), is a vector of attributes of the \( ith \) individual, and the \( jth \) column, denoted by \( \vec{x}_j \), summarizes the distribution of the \( jth \) attribute among \( n \) individuals. Let \( D = \bigcup_{n \in \mathbb{N}_+} \bigcup_{k \in \mathbb{N}_+} M_+(n,k) \) be the set of all such matrices. For each \( X \in D \) and each attribute \( j \), \( \mu_j(X) \) represents the mean value of the \( jth \) attribute and we denoted \( \underline{\mu}(X) = (\mu_1(X), \ldots, \mu_k(X)) \) the vector of the means of attributes.
A multidimensional inequality index is defined as a function $I : D \to \mathbb{R}_+$. In this paper, we assume that $I$ possesses the four following properties, which are straightforward generalizations of their familiar one-dimensional equivalents:

i) **Continuity**: $I$ is a continuous function.

ii) **Anonymity**: $I(X) = I(PX)$, for all $P \in M(n;n)$ permutation matrix.

iii) **Normalization**: $I(X) = 0$ if all rows of the matrix $X$ are identical, i.e., the distribution of each attribute is perfectly equal.

iv) **Replication Invariance**: $I(Y) = I(X)$ if $Y$ is obtained from $X$ by a replication.

The above axioms are insufficient to guarantee that function $I$ be able to capture the essence of multidimensional inequality and to establish whether one multidimensional distribution is more unequal than another. The well-known Pigou-Dalton transfer principle is the basic axiom to order unidimensional distributions in terms of inequality but there is no a unique generalization to the multidimensional framework of this axiom. Some majorization criteria are widely used in the literature to partially order matrices by the degree of inequality. Two of them are used in this paper. They are listed below:

v) **Strict Schur-Concavity**: $I(BX) < I(X)$ for all $B \in M_n(n,n)$ bistochastic that are not permutation matrices.

Atkinson and Bourguignon [2] point out that a multidimensional inequality measure should also be sensitive to the cross-correlation between inequalities in different dimensions. This idea is captured by Tsui [17] who introduces a new majorization criterion based on the kind of transfers defined by Boland and Proschan [4] as correlation increasing transfers: A distribution $Y$ may be derived from a distribution $X$ by a correlation increasing transfer if for some rows indices $p$ and $q$, $p < q$:

$\forall m \neq p, q$ where

$\underline{x}_p \land \underline{x}_q = \left(\min\{x_{p1}, x_{q1}\}, ..., \min\{x_{pk}, x_{qk}\}\right)$

$\overline{x}_p \lor \overline{x}_q = \left(\max\{x_{p1}, x_{q1}\}, ..., \max\{x_{pk}, x_{qk}\}\right)$

Tsui [17] formally introduces the Correlation Increasing Principle as follows:

vi) **Correlation Increasing Principle**: $I(X) < I(Y)$, whenever $Y$ may be derived from $X$ by a permutation of rows and a finite sequence of correlation increasing transfers at least one of which is strict.
The Correlation Increasing Principle has an appealing and intuitive interpretation. We may imagine the situation in which the first individual in the society receives the lowest amount of each attribute; the second individual is endowed with the second lowest amount, up to the individual which receives the greatest amount of each attribute. The Correlation Increasing Principle ensures what seems compelling: this distribution is the most unfair in the sense that any other distribution matrix of the same amount of attributes is less unequal than it.

If the population in which we want to measure inequality is split into groups according to characteristics such as age, gender, race or area of residence, it seems desirable to demand some properties which allow us to relate inequality in each group to overall inequality. A minimal requirement is to demand that if inequality in one group increases, the overall inequality should also increase. This property proposed by Shorrocks [15] in the unidimensional framework is generalized for multidimensional distributions in the following way:

vii) **Aggregative Principle**: An index $I$ is said to be aggregative if there exists an "aggregator" function $A$ such that

$$ I(X) = A(I(X_1, \mu(X_1), n_1), I(X_2, \mu(X_2), n_2) ), \text{ for all } X_1, X_2 \in D \text{ and } A \text{ is continuous and strictly increasing in the index values } I(X_1) \text{ and } I(X_2). $$

In the literature on inequality indices, invariance properties are often invoked.

viii) **Scale Invariance Principle**: $I(X) = I(X \Lambda)$, for all $\Lambda \in \mathbb{M}_+(k,k) /

$$ \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k) \text{ with } \lambda_j > 0. $$

Relative multidimensional inequality indices are those that are scale invariant.

ix) **Translation Invariance Principle**: $I(X) = I(X + A)$, for all $A \in \mathbb{M}_+(n,k)$ with identical rows $a = (a_1, a_2, \ldots, a_k)$ and $a_j \geq 0$.

Absolute multidimensional inequality indices are those that are translation invariant.
3. Multidimensional Unit-Consistency Measures.

The above section ends with two possible answers as to how to distribute a given amount of attributes among all the individuals without altering inequality level. As already mentioned, in the unidimensional framework Zheng [20] has analysed in depth the value judgements involved in the different ways in which this problem is faced and has proposed a new axiom of unit-consistency which requires that the inequality ranking between two distributions should not be affected by the unit in which income is expressed.

In this section we generalize this axiom to the multidimensional framework allowing several attributes to be measured in different units without changing the inequality rankings of the multidimensional distributions. We go on to characterize the class of aggregative multidimensional inequality measures which meet this property.

We propose the following natural generalization of the unit-consistency axiom to the multidimensional framework:

\[ \text{Unit-Consistency Axiom: For any two matrix distributions } X, Y \in M_+^{n,k} \text{ such that } I(X) < I(Y) \text{ then } I(X\Lambda) < I(Y\Lambda) \text{ for any } \Lambda \in M_+^{k,k}/ \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k) \text{ with } \lambda_j > 0. \]

Unit-consistent multidimensional inequality indices are those that are unit-consistent.

Obviously every relative multidimensional inequality measure is unit-consistent. Before formally characterizing the aggregative multidimensional inequality measures which are unit-consistent, it is useful to identify the functional implication of the unit-consistence axiom for a general multidimensional index of inequality. All the proofs are presented in the Appendix.

**Proposition 1:** A multidimensional inequality index \( I : D \rightarrow \mathbb{R}_+ \) is unit-consistent if and only if for any multidimensional distribution \( X \in M_+^{n,k} \) and for any diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k) \) with \( \lambda_j > 0 \), there exists a continuous function \( f : \mathbb{R}_+^k \times \mathbb{R} \rightarrow \mathbb{R} \) increasing in the last argument such that

\[ I(X\Lambda) = f(\lambda_1, \lambda_2, \ldots, \lambda_k, I(X)) \quad (I) \]
This result reveals that in fact if any changes in the attribute units have no influence on inequality rankings, both the unit change matrix $\Lambda$, and the inequality value $I(X)$ must enter into $I(X\Lambda)$ independently.

The main objective of this section, as already mentioned, is to characterize the entire class of unit-consistent aggregative multidimensional inequality measures. The main results of our work are the two following theorems.

**Theorem 2:** A multidimensional inequality measure $I : D \to \mathbb{R}_+$ satisfies Strict Schur-Concavity, the Aggregative Principle and the Unit-Consistency Axiom if and only if there exists a continuous increasing transformation $F : \mathbb{R} \to \mathbb{R}_+$, with $F(0) = 0$, such that for any $X \in M_+(n,k)$ either:

\[
F(I(X)) = \frac{\rho}{n} \prod_{j=1}^{k} \mu_j^{\tau_j} \sum_{i=1}^{n} \left[ \prod_{j=1}^{k} (x_{ij})^{\alpha_j} - \prod_{j=1}^{k} (\mu_j)^{\alpha_j} \right] \tag{2}
\]

where $\tau_j \in \mathbb{R}$ and the parameters $\alpha_j$ and $\rho$ have to be chosen such that the function $\phi(x) = \rho \prod_{j=1}^{k} (x_{ij})^{\alpha_j}$ is strictly convex.

or

\[
F(I(X)) = \frac{1}{n} \prod_{j=1}^{k} \mu_j^{\tau_j} \sum_{i=1}^{n} \left( \sum_{j=1}^{k} a_{mj} \log \frac{x_{ij}}{\mu_j} \right) \tag{3}
\]

where $\tau_j \in \mathbb{R}$ and $m \in \{1,2,...,k\}$ and the parameters $a_{mj}$ have to be chosen such that the function $\phi(x) = \sum_{j=1}^{k} \frac{x_{im}a_{mj}}{U_m} \log(x_{ij})$ is strictly convex.

or

\[
F(I(X)) = \frac{1}{n} \prod_{j=1}^{k} \mu_j^{\tau_j} \sum_{i=1}^{n} \sum_{j=1}^{k} \delta_j \log \left( \frac{\mu_j}{x_{ij}} \right) \tag{4}
\]

where $\tau_j \in \mathbb{R}$ and $\delta_j > 0$ for all $j$. 
As already mentioned the Correlation Increasing Principle is a compelling axiom to order rank matrix distributions in terms of inequality. If this property is also assumed then only the first of these expressions remains with additional conditions upon the coefficients.

**Theorem 3:** A multidimensional inequality measure $I : D \rightarrow \mathbb{R}_+$ satisfies Strict Schur-Concavity, the Correlation Increasing Principle, the Aggregative Principle and the Unit-Consistency Axiom if and only if there exists a continuous increasing transformation $F : \mathbb{R} \rightarrow \mathbb{R}_+$, with $F(0) = 0$, such that for any $X \in M_+ (n,k)$

$$F(I(X)) = \frac{\rho}{n!} \prod_{j=1}^{k} \mu_j^{\alpha_j - \omega} \sum_{\tau} \left[ \prod_{j=1}^{k} (x_{ij})^{\alpha_j} - \prod_{j=1}^{k} (\mu_j)^{\alpha_j} \right]$$

(5)

where $\tau \in \mathbb{R}$, $\rho > 0$, $\alpha_j < 0$, $j = 1,2,...,k$.

**Some remarks about the families derived in Theorem 2 and Theorem 3**

i) Every measure in the family derived in theorem 2, and the subfamily obtained in theorem 3 is aggregative and unit-consistent. In fact, assuming the most usual majorization criteria, what has been proved is that they are the only unit-consistent aggregative multidimensional inequality measures. As already mentioned unit-consistency is a minimal requirement in the sense that it only demands that inequality orderings are not altered when the units in which attributes are measured change. On the other hand, if the population is split into groups, the aggregative principle is also a minimal requirement which only demands that overall inequality should increase if one group inequality increases. Then in empirical applications it makes sense to choose measures from these families.

ii) When we take the transformation $F$ equal to the identity, we find what can be considered “canonical forms” of these unit-consistent measures. As shown in the proofs, these forms fulfil a decomposition property, a sort of generalization of the additive decomposition in the unidimensional framework: for these measures overall inequality can be expressed as the sum of the inequality level of a hypothetical distribution in which each person’s attributes are replaced by the corresponding means of their group and a weighted sum of the group inequality levels.
For these canonical forms it holds that \( I(X\Lambda) = \left(\prod_{1 \leq i \leq k} \lambda_i \right)^\tau I(X) \). As a consequence they are relative measures if and only if \( \tau = 0 \). These cases correspond exactly with the two families which Tsui [17] characterises in Theorems 3 and 4. In other words, the families obtained in this paper are extensions of the two respective classes derived by Tsui [17].

Furthermore, an extreme rightist view holds when \( \tau < 0 \), since in these cases inequality decreases when any attribute is increased for all people by the same proportion. In contrast, when \( \tau > 0 \) in these same situations inequality increases. These measures represent points of view designated as “variable views” according to Amiel and Cowell [1] since the value judgements represented by these measures can vary from the intermediate to the extreme leftists depending on different distributions.

As regards absolute measures, Tsui [16] generalizes the absolute inequality measures proposed by Kolm [9] to the multidimensional setting and characterises the derived family from the social welfare approach. None of these absolute measures is included in the families of Theorems 2 and 3. In addition, it can be proved that there are no unit-consistent absolute aggregative inequality measures fulfilling both Strict Schur-Concavity and the Correlation Increasing Principle ¹.

4. CONCLUDING REMARKS

In this paper we have extended the unit consistency axiom proposed by Zheng to the multidimensional context and have characterised the class of aggregative multidimensional inequality measures which meet this property. We have also identified which measures among them fulfil the Correlation Increasing Principle.

The family of multidimensional inequality measures characterised in Theorem 2, and the subfamily derived in Theorem 3 are extensions of the two respective classes derived by Tsui [17]. As a result the two families identified in this paper contain relative measures, in fact all the relative aggregative multidimensional measures belong to them.

¹ In fact it can be proved that given \( A \in M_n(n,k) \) with identical rows \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \) and \( a_j \geq 0 \), then
\[
\frac{\partial I(X + A)}{\partial a_j} \bigg|_{a_j = 0} = 0
\]
if and only if
\[
\tau \prod_{i < j \in \Omega} \mu_{ij} = \frac{1}{n} \sum_{i < j \in \Omega} \prod_{i < j \in \Omega} x_{ij} \frac{\mu_{ij} a_j - x_j (a_i - \tau)}{x_{ij}},
\]
but this is impossible since the right side term, taking into account that \( \alpha_j < 0 \) tends to infinite when \( x_j \) tends to 0 whereas the left side term is a constant.
We have also shown that no absolute aggregative inequality measures exist which are unit-consistent and fulfil Strict Schur-Concavity and the Correlation Increasing Principle besides. On the other hand, many inequality measures which represent various value judgements are included in these families.

In empirical applications concerned with the measure of inequality in a population classified into groups, both the aggregative principle and the unit-consistency are minimal requirements for an inequality measure. The families derived in this paper meet both properties and allow us to adopt different value judgements in measuring multidimensional inequality.
REFERENCES


APPENDIX

Proof of Proposition 1: For any $X \in M_+ (n,k)$ and for any $\Lambda \in M_+ (k,k)$, with $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$, we define $J(X) = I(X\Lambda)$. The unit-consistency axiom implies that if $I(X) = I(Y)$ then $I(X\Lambda) = I(Y\Lambda)$, i.e., $J(X) = J(Y)$. Moreover, it also implies that if $I(X) < I(Y)$ then $J(X) < J(Y)$. As a result it follows that $J(X)$ is an increasing function in $I(X)$. Hence, there exists an increasing function $f_{\lambda_1, \lambda_2, \ldots, \lambda_k} : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$J(X) = f_{\lambda_1, \lambda_2, \ldots, \lambda_k}(I(X)) \quad (6)$$

Since both $J(X)$ and $I(X)$ are continuous functions of $X$, it follows that $f_{\lambda_1, \lambda_2, \ldots, \lambda_k}(\cdot)$ is also a continuous function. Defining $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ by $f(\lambda_1, \lambda_2, \ldots, \lambda_k ; I) = f_{\lambda_1, \lambda_2, \ldots, \lambda_k}(\cdot)$ we have

$$I(X\Lambda) = J(X) = f(\lambda_1, \lambda_2, \ldots, \lambda_k ; I(X)) \quad (7)$$

where $f(\lambda_1, \lambda_2, \ldots, \lambda_k ; I(X))$ is also a continuous function in the first arguments $\lambda_1, \lambda_2, \ldots, \lambda_k$. Indeed, for any $j=1,\ldots,k$, infinitesimal changes in $\lambda_j$ produce simultaneous infinitesimal changes in the $\lambda_j x_i$s. Therefore, since $I$ is a continuous function, they also produce small changes in $I(X\Lambda)$, and, as a consequence, $f$ is continuous in $\lambda_j$, which completes the proof of the necessity. The sufficiency of the proposition is straightforward.

$Q.E.D.$

In order to prove theorem 2 and consequently the particular situation considered in theorem 3, we follow two steps. Firstly we get a characterization theorem for a subfamily which meets a sort of decomposition property which demands that overall inequality can be expressed as the sum of the inequality level of a hypothetical distribution in which each person’s attributes are replaced by the corresponding means of their group and a weighted sum of the group inequality levels. Then, following the equivalent unidimensional, we show that every aggregative measure can be expressed as an increasing transformation of one member of this family.
Let’s begin with a previous definition and some results.

**Definition 4:** If any population is classified in $G$ non-empty subgroups $X = (X_1, X_2, ..., X_G)$, decomposability requires the following relationship between the total inequality value $I(X)$ and the subgroup inequality values $I(X_g)$:

$$I(X) = I(X_1, X_2, ..., X_G) = \sum_{g=1}^{G} w_g \left( \mu(X_g), n(X_g) \right) I(X_g) + I(A_{\Lambda_1}, ..., A_{\Lambda_G})$$

where $w_g$ is the weight attached to subgroup $g$, $A_g \in \left\{ n(X_g), k \right\}$ of 1’s and $\Lambda_g \in M_{g}(k,k)/\Lambda_g = \text{diag} \left( \mu_1(X_g), \mu_2(X_g), ..., \mu_k(X_g) \right)$ for $g=1, ..., G$.

**Lemma 5:** If a multidimensional inequality measure $I: D \to \mathbb{R}_+$ satisfies **Strict Schur-Concavity, Decomposability and the Unit-Consistency Axiom**, then

$$I(X\Lambda) = \left( \prod_{1 \leq i < j \leq k} \lambda_{ij} \right)^{\tau} I(X)$$

for any $X \in M_{k}(n,k)$ and $\Lambda \in M_{k}(k,k)/\Lambda = \text{diag} \left( \lambda_1, \lambda_2, ..., \lambda_k \right)$ with $\lambda_j > 0$, and some constant $\tau \in \mathbb{R}$.

Moreover $I$ is a homogenous function of degree $k\tau$.

**Proof:** (Following Shorrocks [15] and Zheng [20]). For any multidimensional distribution $X \in M_{k}(n,k)$ let $w(X) = \left( \mu(X), n(X) \right) = (\mu, n) \in \mathbb{R}_{++}^{k+1}$ be a “parameter-vector” for the distribution $X$.

The set of $X \in D$ with a common parameter-vector $w$, constitutes the set $S(w) = \left\{ X \in D / w(X) = w \right\}$. For each $w$, $S(w)$ is a connected, open subset of $D$ containing more than one element. Hence, by continuity, normalization and strict Schur-concavity

$$I(S(w)) = \left\{ I(X) / X \in S(w) \right\} = \left[ 0, \xi(w) \right]$$

where $\xi(w)$ is strictly positive and may be finite and infinite.

Define $\Omega = \left\{ w(X) / X \in D \right\}$. For each $w = (\mu, n) \in \Omega$ let $X$ and $Y$ be any two distributions with a common parameter vector $w$. By definition, $\mu(X) = \mu(Y) = \mu$ and
Now consider a new distribution $Z = (X, Y)$. Since $I$ is a decomposable measure, we have

$$I(Z) = w_1(\mu, n)I(X) + w_2(\mu, n)I(Y)$$

(9)

where $w_1(\mu, n)$ and $w_2(\mu, n)$ are the weights for distributions $X$ and $Y$ respectively. The between-group inequality term in (9) is equal to 0 since $I$ satisfies the normalization principle. Note also that $\mu(Z) = \mu$ and $n(Z) = 2n$.

Now multiplying the distributions $X$, $Y$ and $Z$ by any $\Lambda \in M_+(k, k)$ with $\lambda_j > 0$ we have

$$I(Z\Lambda) = w_1(\mu\Lambda, n)I(X\Lambda) + w_2(\mu\Lambda, n)I(Y\Lambda)$$

(10)

Assuming that $I$ is unit-consistent and taking into account the proposition 1 there exists a continuous function which is increasing in the last argument, $f: \mathbb{R}_{+}^k \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(\lambda_1, \lambda_2, \ldots, \lambda_k; I(Z)) = w_1(\mu, n)f(\lambda_1, \lambda_2, \ldots, \lambda_k; I(X)) + w_2(\mu, n)f(\lambda_1, \lambda_2, \ldots, \lambda_k; I(Y))$$

(11)

Substituting (9) into (11) we further have

$$f(\lambda_1, \lambda_2, \ldots, \lambda_k; w_1(\mu, n)I(X) + w_2(\mu, n)I(Y)) =$$

$$= w_1(\mu\Lambda, n)f(\lambda_1, \lambda_2, \ldots, \lambda_k; I(X)) + w_2(\mu\Lambda, n)f(\lambda_1, \lambda_2, \ldots, \lambda_k; I(Y))$$

(12)

Denoting $f(\lambda_1, \lambda_2, \ldots, \lambda_k; ) = \tilde{f}(\cdot)$, $I(X) = K$, $I(Y) = L$, $w_g(\mu, n) = w_g$ and $w_g(\mu, n) = \tilde{w}_g$ for $g=1,2$, equation (12) can be rewritten

$$\tilde{f}(w_1K + w_2L) = \tilde{w}_1\tilde{f}(K) + \tilde{w}_2\tilde{f}(L)$$

(13)

for all $K, L \in [0, \xi(w)]$. The solution to this functional equation (Aczél [3], p.66) is

$$w_1 = \tilde{w}_1, \; w_2 = \tilde{w}_2$$

and

$$\tilde{f}(K) = \alpha K$$

(14)

for some constant $\alpha \neq 0$.

That is

$$I(X\Lambda) = f(\lambda_1, \lambda_2, \ldots, \lambda_k; I(X)) = \tilde{f}(I(X)) = \alpha(\lambda_1, \lambda_2, \ldots, \lambda_k)I(X)$$

Simplifying we write
for any $X \in M_+(n,k)$ and $\Lambda \in M_+(k,k) / \Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_k)$ with $\lambda_j > 0$ and some positive function $\alpha(\cdot)$.

The proof is completed by noting that for any two matrices $\Lambda, H \in M_+(k,k) / \Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_k)$ and $H = \text{diag}(h_1, h_2, ..., h_k)$ with $\lambda_j$ and $h_j$ real and positive numbers, and from (15) we have

$$\alpha(\Lambda H) = \alpha(\Lambda) \alpha(H)$$

and the solution to this functional equation (Aczél ([3], p.350)) is

$$\alpha(\Lambda) = \alpha(\lambda_1, \lambda_2, ..., \lambda_k) = |\text{det}(\Lambda)| = (\lambda_1 \lambda_2 ... \lambda_k)^{\tau}$$

where $\tau$ is an arbitrary real constant and $|\text{det}(\Lambda)|$ is the determinant of $\Lambda$, concluding that

$$I(X\Lambda) = (\lambda_1 \lambda_2 ... \lambda_k)^{\tau} I(X)$$

for any $X \in M_+(n,k)$ and $\tau \in \mathbb{R}$.

Let’s see that $I$ is a homogenous function of degree $k \tau$. For all $t \in \mathbb{R}_+$, $I(tX) = I(Xt) = \alpha(t, t, ..., t) I(X) = |\text{det}(T)|^{\tau} I(X) = t^{k\tau} I(X)$

where $T \in M_+(k,k) / T = \text{diag}(t, t, ..., t)$ with $t > 0$.

Q.E.D.

**Lemma 6**: A multidimensional inequality measure $I: D \rightarrow \mathbb{R}_+$ satisfies Strict Schur-Concavity, Decomposability and the Unit-Consistency Axiom if and only it is a positive multiple of the form

$$I(X) = \rho \frac{1}{n!} \prod_{j=1}^{k} \left[ \prod_{i=1}^{n} (x_{ij})^{a_{ij}} - \prod_{j=1}^{k} (\mu_j)^{a_{ij}} \right]$$

where $\tau \in \mathbb{R}$ and the parameters $\alpha_j$ and $\rho$ have to be chosen such that the function

$$\phi(\Xi) = \rho \prod_{j \in \mathcal{A}_k} (x_{ij})^{a_j}$$

is strictly convex.
\[ I(X) = \frac{1}{n} \prod_{j=1}^{k} \mu_j^{-\tau} \left( \sum_{i=1}^{n} \frac{x_{im}}{\mu_i} \right) \left[ \sum_{j=1}^{k} a_{mj} \log \left( \frac{x_{ij}}{\mu_j} \right) \right] \]  

(17)

where \( \tau \in \mathbb{R} \) and \( m \in \{1,2,...,k\} \) and the parameters \( a_{mj} \) have to be chosen such that

the function \( \phi(x) = \sum_{j=1}^{k} \frac{x_{im}a_{mj}}{u_m} \log(x_{ij}) \) is strictly convex.

or

\[ I(X) = \frac{1}{n} \prod_{j=1}^{k} \mu_j^{-\tau} \left( \sum_{i=1}^{n} \delta_j \log \left( \frac{\mu_j}{x_{ij}} \right) \right) \]  

(18)

where \( \tau \in \mathbb{R} \) and \( \delta_j > 0 \) for all \( j \).

Proof: If \( I \) satisfies strict Schur-concavity, continuity, normalization, the aggregative principle and the replication invariance principle, Tsui ([17], Theorem 1) establishes there exist continuous functions \( \phi \) and \( F \) such that, for every \( X \in \mathcal{M}_+(n,k) \) with mean vector \( \mu(X) = (\mu_1(X),...,\mu_k(X)) \) we get

\[ F(I(X),\mu) = \frac{1}{n} \sum_{\tau=1}^{n} \left( \phi(\tau) - \phi(\mu) \right) \]  

(19)

where \( F \) is strictly increasing in \( I(X) \), \( F(0,\mu) = 0 \) and \( \phi \) is strictly convex, which specifies the structure of aggregative multidimensional inequality measures.

Now consider the same distributions \( X, Y \) and \( Z = (X,Y) \), as they were considered in the proof of lemma 5, that is, \( \mu(X) = \mu(Y) = \mu \) and \( n(X) = n(Y) = n \). Since all decomposable multidimensional inequality measure is also aggregative applying (19) and the decomposability of \( I \) we have

\[ F(I(Z),\mu) = F(w_1(\mu,n)I(X) + w_2(\mu,n)I(Y),\mu) = 0.5F(I(X),\mu) + 0.5F(I(Y),\mu) \]  

(20)

Denote \( F(.,\mu) = \bar{F}(.) \), \( I(X) = K \), \( I(Y) = L \), \( w_g(\mu,n) = w_g \) for \( g=1,2 \). Then we can rewrite (20) as follows

\[ \bar{F}(w_1K + w_2L) = 0.5\bar{F}(K) + 0.5\bar{F}(L) \]  

(21)
for all $K, L \in [0, \xi(w)]$. Resorting to Aczél ([3], p.66) once again, the solution to (21) also satisfies

$$\tilde{F}(K + L) = \tilde{F}(K) + \tilde{F}(L)$$

(22)

whose nontrivial solution is

$$\tilde{F}(K) = \lambda K$$

for some constant $\lambda \neq 0$

(23)

Replacing in (23) $\tilde{F}(.)$ with $F(., \mu)$, $K$ with $I(X)$ and using (19) we have

$$I(X) = \frac{1}{n \lambda(\mu)} \sum_{i=1}^{n} \left( \phi(x_i) - \phi(\mu) \right)$$

(24)

for some continuous function $\lambda(.)$.

By the lemma 5 since $I$ satisfies strict Schur-concavity, decomposability and unit-consistency, then $I$ is a homogenous function of degree $k \tau$.

Let’s define

$$G(X) = I(X) \prod_{j=1}^{k} \mu_j^\tau = \frac{I(X)}{\prod_{j=1}^{k} \mu_j^\tau}$$

(25)

with $\tau \in \mathbb{R}$.

Since $I$ is a decomposable measure, it is easy to see that $G(X)$ is also decomposable and therefore aggregative. Moreover $G(X)$ is homogenous of degree zero, that is, $G(X)$ satisfies the scale-invariance principle, since for any $t \in \mathbb{R}_{++}$, taking into account that $I$ is homogeneous of degree $k \tau$, we get

$$G(tX) = G(XT) = I(XT) / \prod_{j=1}^{k} (t \mu_j)^\tau = \frac{t^{k \tau} I(X)}{t^{k \tau} \prod_{j=1}^{k} \mu_j^\tau} = G(X),$$

where $T \in M_{+}(k,k) / T = \text{diag}(t, t, \ldots, t)$ with $t > 0$.

Applying the first functional expression in Tsui ([17], Theorem 3) to $G(X)$ there exists a transformation $F$ such that, for any $X \in M_{+}(n,k)$ with mean vector $\mu$, we get

$$F(G(X)) = F \left( I(X) / \prod_{j=1}^{k} \mu_j^\tau \right) = \frac{\rho}{n} \sum_{i=1}^{n} \left[ \prod_{j=1}^{k} \left( \frac{x_{ij}}{\mu_j} \right)^{a_j} - 1 \right]$$

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$$\frac{\rho}{n_k}\prod_{j=1}^{n_k} \mu_j^{a_j} \sum_{i=1}^{n_k} \left( \prod_{j=1}^{k} x_{ij}^{a_j} - \prod_{j=1}^{k} \mu_j^{a_j} \right)$$  \hspace{1cm} (26)$$

where \( \sum_{i \leq j} \text{sgn}(\sigma) \prod_{i \leq j} \rho \alpha_{in(i)} > 0 \); \( \alpha_{in(i)} = \alpha_i \alpha_{n(i)} \) if \( i \neq \sigma(i) \); \( \alpha_{in(i)} = \alpha_i (\alpha_i - 1) \) if \( i = \sigma(i) \). \( \zeta_j \) denotes the set of permutations of \( \{1,2,...,j\} \) \( \forall j \in K \), \( \text{sgn}(\sigma) = -1 \) if the permutation is odd.

The proof of Theorem 3 by Tsui shows that these conditions upon the coefficients are in fact equivalent to demand that the function \( \phi(x) = \rho \prod_{j \leq k} (x_j)^{\alpha_j} \) be strictly convex.

Now let’s consider the same distributions \( X, Y \) and \( Z = (X,Y) \), as they were considered in the proof of the lemma 5, that is, \( \mu(X) = \mu(Y) = \mu \) and \( n(X) = n(Y) = n \). Applying equations (25), (26) and the decomposability of \( G \) we have

$$ F \left( w_1 (\mu, n) \frac{I(X)}{\prod_{j=1}^{k} \mu_j^\sigma} + w_2 (\mu, n) \frac{I(Y)}{\prod_{j=1}^{k} \mu_j^\sigma} \right) = 0.5F \left( \frac{I(X)}{\prod_{j=1}^{k} \mu_j^\sigma} \right) + 0.5F \left( \frac{I(Y)}{\prod_{j=1}^{k} \mu_j^\sigma} \right)$$  \hspace{1cm} (27)$$

Denoting \( I(X) = K \), \( I(Y) = L \), \( w_g (\mu, n) = w_g \) for \( g = 1,2 \), equation (27) becomes

$$ F \left( w_1 \frac{K}{\prod_{j=1}^{k} \mu_j^\sigma} + w_2 \frac{L}{\prod_{j=1}^{k} \mu_j^\sigma} \right) = 0.5F \left( \frac{K}{\prod_{j=1}^{k} \mu_j^\sigma} \right) + 0.5F \left( \frac{L}{\prod_{j=1}^{k} \mu_j^\sigma} \right)$$  \hspace{1cm} (28)$$

for all \( K, L \in \left[ 0, \xi(w) \right] \). Resorting to Aczél ([3], p.66) once again, we know that the solution to (28) also satisfies

$$ F \left( \frac{K}{\prod_{j=1}^{k} \mu_j^\sigma} + \frac{L}{\prod_{j=1}^{k} \mu_j^\sigma} \right) = F \left( \frac{K}{\prod_{j=1}^{k} \mu_j^\sigma} \right) + F \left( \frac{L}{\prod_{j=1}^{k} \mu_j^\sigma} \right)$$  \hspace{1cm} (29)$$

whose nontrivial solution is
\[
F \left( \frac{K}{\prod_{j=1}^{k} \mu_j^\tau} \right) = \rho \frac{K}{\prod_{j=1}^{k} \mu_j^\tau} \quad \text{for some constant } \rho \neq 0 \quad (30)
\]

Substituting onto (26) and replacing \( K \) with \( I(X) \) we have that \( I \) is a positive multiple of the form

\[
I(X) = \frac{\rho}{n} \frac{\prod_{j=1}^{k} \mu_j^{\alpha_j}}{\prod_{j=1}^{k} \mu_j^{\tau}} \sum_{i=1}^{n} \left[ \prod_{j=1}^{k} (x_{ij})^{\alpha_j} - \prod_{j=1}^{k} (\mu_j)^{\alpha_j} \right]
\]

where \( \tau \in \mathbb{R} \) and the parameters \( \alpha_j \) and \( \rho \) have to be chosen such that the function \( \phi(x_j) = \rho \prod_{j \in \mathbb{R}} (x_{ij})^{\alpha_j} \) is strictly convex.

In a similar way we can derive the other functional forms (17) and (18) considering the other two functional expression in Tsui ([17], Theorem 3), which completes the proof of the necessity of the lemma.

As regards the sufficiency of the lemma, it is easy to see that the functional forms (16), (17) and (18) are decomposable with weights respectively

\[
w_g = \frac{n_g}{n} \prod_{j=1}^{k} \left( \frac{\mu_j(X_g)}{\mu_j} \right)^{\alpha_j}, \quad w_g = \frac{n_g}{n} \prod_{j=1}^{k} \left( \frac{\mu_j(X_g)}{\mu_j} \right)^{\tau}, \quad w_g = \frac{n_g}{n} \prod_{j=1}^{k} \left( \frac{\mu_j(X_g)}{\mu_j} \right)^{\rho}
\]

for all \( g=1,...,G \)

It is also straightforward to prove that these three forms satisfy strict Schur-concavity, continuity and normalization.

The sufficiency of the lemma is completed proving that these three functional forms are unit-consistent.

We are going to prove that the first functional form is unit-consistent, in the same way we can conclude for the other functional forms.

For any \( X \in M_{+}(n,k) \) and \( \Lambda \in M_{+}(k,k) / \Lambda = \text{diag}(\lambda_1,\lambda_2,...,\lambda_k) \) with \( \lambda_j > 0 \).

\[
I(X\Lambda) = \frac{\rho}{n \prod_{j=1}^{k} (\lambda_j \mu_j)^{\tau}} \sum_{i=1}^{n} \left[ \prod_{j=1}^{k} \left( \frac{\lambda_j x_{ij}}{\lambda_j \mu_j} \right)^{\alpha_j} - 1 \right] = \frac{\rho}{n (\lambda_1 \lambda_2 ... \lambda_k)^{\tau} \prod_{j=1}^{k} \mu_j^{\tau}} \sum_{i=1}^{n} \left[ \prod_{j=1}^{k} \left( \frac{x_{ij}}{\mu_j} \right)^{\alpha_j} - 1 \right] = \left( \lambda_1, \lambda_2 ... \lambda_k \right)^{\tau} I(X)
\]
Thus there exists a continuous function $f$ which is increasing in the last argument, such that

$$I(X\Lambda) = f(\lambda_1, \lambda_2, \ldots, \lambda_k; I(X))$$

After proposition 1 $I$ is unit-consistent.

Q.E.D.

**Lemma 7:** A multidimensional inequality measure $I : D \rightarrow \mathbb{R}_+$ satisfies Strict Schur-Concavity, the Correlation Increasing Principle, Decomposability and the Unit-Consistency Axiom if and only if it is a positive multiple of the form

$$I(X) = \frac{\rho}{n} \prod_{j=1}^k \mu_j^{\alpha_j} \sum \left[ \prod_{j=1}^k (x_{ij})^{\alpha_j} - \prod_{j=1}^k (\mu_j)^{\alpha_j} \right]$$

(31)

where $\tau \in \mathbb{R}$, $\rho > 0$, $\alpha_j < 0$, $j = 1, 2, \ldots, k$.

**Proof:** The proof is straightforward following Tsui ([17], Theorem 4) it can be proved that the last two functional forms given by equations (17) and (18) of the lemma 6 are incompatible with the correlation increasing axiom.

Moreover the correlation increasing axiom requires that $\phi$ defined in the same as in the previous proof should be not only strictly convex but also strictly L-superadditive. Hence we can clarify the restrictions on the parameters, which reduce to $\rho > 0$, $\alpha_j < 0$, $j = 1, 2, \ldots, k$.

Q.E.D.

**Proof of Theorem 2:** One can easily adapt the results in Shorrocks [15] to show that for any continuous aggregative multidimensional inequality index $J$ there exists a decomposable multidimensional inequality index $I$ and a continuous strictly increasing function $G : \mathbb{R} \rightarrow \mathbb{R}$ with $G(0) = 0$ such that

$$I(X) = G(J(X))$$

Moreover, if $J$ is unit-consistent the same holds for $I$. Indeed, if $I(X) < I(Y)$ i.e. $G(J(X)) < G(J(Y))$ since $G$ is a strictly increasing function then $J(X) < J(Y)$. As a consequence, for any $X \in M_+ (n,k)$ and $\Lambda \in M_+ (k,k)$ $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$ with
If \( \lambda_j > 0 \), we have \( J(X\Lambda) < J(Y\Lambda) \) and then
\[
G(J(X\Lambda)) < G(J(Y\Lambda)) \text{i.e. } I(X\Lambda) < I(Y\Lambda),
\]
concluding that \( I \) a unit-consistent multidimensional inequality index.

Denoting \( F = G^{-1} \), we have that if \( J \) satisfies strict Schur-concavity, the aggregative principle and the unit-consistency axiom, there exists a continuous function \( F \) such that, for every \( X \in M_+(n,k) \)
\[
J(X) = F(I(X))
\]
where \( F \) is strictly increasing and \( I \) is a decomposable and unit-consistent multidimensional inequality index. Therefore \( I \) belongs to the class characterized in lemma 6.

The sufficiency of this theorem is straightforward.

Q.E.D

Proof of Theorem 3: If \( J \) satisfies the correlation increasing principle, since \( F \) is a strictly increasing function, then \( I \) also satisfies the correlation increasing principle. Therefore \( I \) is a multidimensional inequality index which belongs to the class characterized in the lemma 7. This proves the necessity of the theorem. Once again the sufficiency of this theorem is straightforward.

Q.E.D