Constructing indices of multivariate polarization*

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November 7, 2006

Abstract

Multivariate indices of polarization are constructed to measure effects of non-income attributes like wealth and education. Adopting the view of Esteban and Ray (1994), polarization is considered as the presence of groups which are internally homogeneous, externally heterogeneous, and of similar size. We propose a class of polarization indices which is built from measures of relative groups size and from decomposable indices of socio-economic inequality. For the latter, we employ the special inequality indices of Maasoumi (1986), Tsui (1995, 1999) and Koshevoy and Mosler (1997). Then, postulates for multidimensional polarization measurement are stated and discussed. The approach is illustrated by a numerical example.

Keywords: Polarization index, decomposable inequality indices, multidimensional inequality, multivariate social evaluation.

JEL= D63, C43.

1 Introduction

Polarization is commonly connected with the division of a society into groups as possible cause of social conflicts. It is measured by quantifying and comparing socio-economic disparity, not only in terms of differences among individuals (as inequality measurement does) but also in terms of differences among population groups.

The first systematic investigations into indices and postulates of polarization measurement are due to Wolfson (1994, 1997) and Esteban and Ray (1994). These

*We thank Christoph Scheicher, Satya R. Chakravarty and Esfandiar Maasoumi for useful remarks.

pioneering papers have been followed by many others, among them Chakravarty and Majumder (2001), Esteban et al. (1999), Wang and Tsui (2000), D’Ambrosio (2001), Gradín (2000), Duclos et al. (2004). All these papers study polarization in terms of the distribution of incomes and measure how much this distribution spreads out from its center, dividing the population into at least two groups that are homogeneous and well separated from each other.

In case of two groups, the phenomenon can be also seen as a decline of the central part of the distribution. Correspondingly, two strands are distinguished in the literature on univariate polarization: the first one, going back to Wolfson (1994), describes the decline of the middle class, measuring how the center of the income distribution is emptied. The second strand, originating from Esteban and Ray (1994), focuses on the rise of separated income groups; polarization is the larger the more homogeneous the groups are, the more separate, and the more equal in size.

But, societal status of a person and distance between persons (and groups) is not determined by income alone but also by other monetary and non-monetary characteristics of well-being, such as wealth, education, and health. In the measurement of economic inequality and poverty, several authors have pointed out that attributes beyond income should be included in the analysis. Consequently, they have introduced various multi-attribute measures of inequality and poverty; see Atkinson and Bourguignon (1982), Kolm (1977), Maasoumi (1986), Bourguignon and Chakravarty (2003), Tsui (1995, 1999).

Obviously, also the splitting of a society into groups is influenced by attributes besides income. Davis and Huston (1992) have investigated the causes of lower and upper class membership by regression on many socio-economic attributes. But, to our knowledge, there exists no attempt in the literature to measure polarization in many attributes. This paper presents a first inquiry into the multi-attribute measurement of polarization. Our approach follows the second strand of literature: multi-attribute polarization corresponds to splitting the society into groups that are well separated, inside homogeneous, and of comparable size.

We construct multivariate indices of polarization, using the decomposition by subgroups of certain indices of multivariate inequality. These indices can be decomposed into a ‘within groups’ component and a ‘between groups’ component of inequality. Based on them we introduce multivariate polarization indices that increase with respect to between groups inequality and decrease with respect to within groups inequality. Besides, the relative size of the groups matters. Therefore, we employ simple measures of relative groups size that indicate the deviation from equally sized groups and construct polarization indices which, additionally, decrease in these measures. Thus, our approach results in indices which are function of three elements: the measures of inequality between groups, of inequality within groups and of relative groups size.

Such indices apply to arbitrary grouped data and do not assume given groups or

relative group sizes.

Further, classical postulates on the measurement of univariate polarization are considered and extended to the multivariate setting. We then investigate how these postulates are satisfied by our polarization indices.

Two particular problems are intrinsic to the multivariate setting: First, while with income alone people naturally divide into two groups above and below the center, with more than one attribute an infinity of directions arise that point away from the center. Second, in the evaluation of a multiattribute distribution, possible interactions between the attributes have to be modelled. E.g., two attributes may be taken either as substitutes or as complements.

The paper is organized as follows: in Section 2 the general principle of construction is introduced, including special measures of groups size. Sections 3 and 4 study special indices of multiattribute inequality, their decompositions, and the polarization indices built on them. In Section 5 we provide postulates on the measurement of multiattribute polarization and investigate how they are satisfied by our special indices. Section 6 is devoted to a discussion of value interaction among attributes. In Section 7 an illustrative numerical example is given. Section 8 concludes.

2 Index construction

Consider a population of N individuals and their endowments in K attributes. The distribution is notated by a matrix $\mathbf{X}$,

$$\mathbf{X} = \begin{bmatrix} x_{11} & \dots & x_{1K} \\ \vdots & \ddots & \vdots \\ x_{N1} & \dots & x_{NK} \end{bmatrix}_{N \times K}$$

where x_{ik} denotes the endowment of individual i with attribute k . $\mathcal{M}^{N \times K}$ is the set of all $N \times K$ matrices, and $\mathbb{R}_+^K$ is the non-negative orthant of the Euclidean K -space $\mathbb{R}^K$. The row $\mathbf{x}_i = (x_{i1}, \dots, x_{iK})$ represents the endowment of the i -th individual, $i = 1, \dots, N$, while the column $\mathbf{x}^k = (x_{1k}, \dots, x_{Nk})^T$ describes the distribution of the k -th variable, $k = 1, \dots, K$. With $\bar{x}_k$ we indicate the mean value of the k -th variable and with $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_K)$ the total mean vector.

2.1 Measuring polarization through inequality decomposition

As mentioned in the introduction, our concept of polarization is based on the idea that the population divides into groups which are, according to the given attributes,

homogeneous inside and different to each other. Therefore, the more evident are these two phenomena, the more polarized is the society. Moreover, the more equal are the sizes of the different groups, the more increases the polarization level.

So, we propose polarization indices which are basically functions of three elements: inequality between groups, inequality within groups, and relative groups size. Given the groups, such an index has the form

$$P(\mathbf{X}) = \zeta(B(\mathbf{X}), W(\mathbf{X}), S(\mathbf{X})), \quad \mathbf{X} \in \mathcal{M}^{N \times K}, \quad (1)$$

where B and W are indices that measure inequality between and within groups, respectively, S is an index of relative groups size, and ζ is a function $\mathbb{R}^3 \rightarrow \mathbb{R}$ that increases on B and S , and decreases on W .

Concerning B , W and S we make the following *normalizing assumptions*: The three measures have infimum zero. S takes its minimum $S = 0$ if there exists just one non-empty group and its maximum $S = 1$ if there are $G \geq 2$ groups of equal size. $W = 0$ if all groups are internally homogeneous, that is, all individuals in a group have the same endowment. B shall be minimal, equal to 0, in absence of intergroup inequality.

2.2 Special index types

Particular forms of the general class of polarization indices in (1) are

$$P_1(\mathbf{X}) = \phi \left(\frac{B(\mathbf{X})}{W(\mathbf{X}) + c} \right) \cdot S(\mathbf{X}), \quad (2)$$

$$P_2(\mathbf{X}) = \psi(B(\mathbf{X}) - W(\mathbf{X})) \cdot S(\mathbf{X}), \quad (3)$$

$$P_3(\mathbf{X}) = \tau \left(\frac{B(\mathbf{X})}{\sigma(B(\mathbf{X}), W(\mathbf{X})) + c} \right) \cdot S(\mathbf{X}). \quad (4)$$

These types of measures will be used later, in Section 3, with additively decomposable inequality indices.

In P_3 , σ denotes a function which increases in both arguments and is concave in the first one. Thus, $I(\mathbf{X}) = \sigma(B(\mathbf{X}), W(\mathbf{X}))$ can be seen as an index of inequality in the entire population. The constant c must be positive and may depend on the choice of B , W and σ . The functions ϕ , ψ and τ are assumed to be continuous and strictly increasing, with $\phi(0) = \tau(0) = 0$. Consequently, P_3 increases strictly with B .

The functions ϕ , ψ and τ will be chosen in a way that the indices P_i become normalized with infimum value 0 and supremum value 1.

Then, in case of groups having null intergroup inequality, indices P_1 and P_3 are minimum and equal to 0, regardless of the value of W . Measure P_2 , instead, is

minimum when intergroup inequality vanishes and inequality within is maximum, with $\bar{W} = \sup W$; an appropriate choice of the function ψ , such that $\psi(-\bar{W}) = 0$, may guarantee the normalization of P_2 .

The maximum value for $P_i, i = 1, 2, 3$, is attained when B and S are maximum and W is minimum. By assumption, the infimum of W is 0 and the supremum of S is 1. In this case,

$$P_1(\mathbf{X}) = \phi\left(\frac{B(\mathbf{X})}{c}\right), \quad P_2(\mathbf{X}) = \psi(B(\mathbf{X})), \quad P_3(\mathbf{X}) = \tau\left(\frac{B(\mathbf{X})}{\sigma(B(\mathbf{X}), 0) + c}\right).$$

Hence, given two or more groups of equal size and internal homogeneity, each of the three polarization indices is maximized if and only if the inequality between groups is maximized.

The multivariate indices P_1, P_2 and P_3 are similar to well known univariate measures of income polarization. For example, Zhang and Kanbur (2001) use the ratio of income inequality between and inequality within groups, like the measure P_1 in (2). Also Wolfson's measure can be rewritten, analogous to P_2 in (3), as a function of the difference between the Gini index G_B between groups and the Gini index G_W within groups,

$$P^W = \frac{2\bar{x}}{m}(G_B - G_W),$$

where $\bar{x}$ is the mean income and m is the median income; see Rodríguez and Salas (2003). Also the index P^{EGR} of Esteban et al. (1999) is the difference between a term of inequality between groups and within groups,

$$\begin{aligned} P^{EGR} &= B^{EGR} - W^{EGR}, \\ B^{EGR} &= \sum_{g=1}^G \sum_{h=1}^G \pi_g^{1+\alpha} \pi_h |\bar{x}^g - \bar{x}^h|, \\ W^{EGR} &= \beta[G_I - G_B], \end{aligned}$$

where G_I is the Gini index of the entire distribution, $\bar{x}^g$ is the mean income of group g , π_g is the population share of group g and the incomes are normalized with the mean, $\alpha, \beta > 0$.

In order to construct polarization measures of type (1), we will first consider measures of groups size. In Section 3 we shall study multivariate measures of inequality that can be additively decomposed by subgroups, obtaining specific polarization measures of the types P_1, P_2 and P_3 . In Section 4 we consider, instead, inequality measures which are decomposable in non-additive ways and for them we propose other particular forms for P in (1).

2.3 Measures of groups size

As already noticed, an important component of our polarization measures is given by the relative size of groups. Also Esteban and Ray (1994) and D'Ambrosio (2001) underline that a polarization index has to register the differences in the frequencies among groups, so that the more similar are the clusters sizes, the more polarized is the population. We need a function which measures how equally populated are the groups, taking maximum value when the groups sizes are identical, and minimum value in case of a very unequal population distribution.

Let us assume that the population is split into G groups and, without loss of generality, let us order them from above by their population size, so that $N_1 \geq N_2 \geq \dots \geq N_G$ and $N = \sum_{g=1}^G N_g$.

In the case of two groups, a simple measure of relative groups size is

$$S_1(\mathbf{X}) = 1 - \left| \frac{N_1}{N} - \frac{N_2}{N} \right|. \quad (5)$$

The index S_1 is equal to one if the two groups are equally sized. Its value approaches zero if one of the groups becomes empty.

More generally, for $G \geq 2$, we propose *relative groups size measures* that are inverse concentration measures, more specifically, that are functions of weighted means of the relative groups sizes,

$$S(\mathbf{X}) = \gamma \left(\sum_{g=1}^G h_g \frac{N_g}{N} \right), \quad (6)$$

with $\gamma : \mathbb{R}_+ \rightarrow [0, 1]$ strictly monotone and continuous, and $h_g \geq 0$.

We consider here three of the most common concentration indices of the type in (6), i.e. the Herfindahl, Rosenbluth and negative entropy indices¹. They reach minimum value ($= 1/G$ for Herfindahl and Rosenbluth indices and $= -\log G$ for the negative entropy) if the population is split into groups of equal size, and maximum value ($= 1$ for Herfindahl and Rosenbluth indices and $= 0$ for the negative entropy²) if the population is made up by just one non-empty group. We choose, therefore, appropriate γ such that the corresponding measures S assume values between 0, for minimum inequality in groups size, and 1, in case of equally sized groups; proposals are shown in Table 1.

¹For details, see Piesch (1975).

²with $0 \cdot \log 0 = 0$.

Table 1: Special indices (6) of relative groups size.

Index based on	$S(\mathbf{X})$	$\gamma(y)$	h_g
Herfindahl	$\frac{G}{G-1} \left(1 - \sum_{g=1}^G \left(\frac{N_g}{N} \right)^2 \right)$	$\frac{G}{G-1} (1 - y)$	$\frac{N_g}{N}$
Rosenbluth	$\frac{G}{G-1} \left(1 - \left(2 \sum_{g=1}^G \frac{N_g g}{N} - 1 \right)^{-1} \right)$	$\frac{G}{G-1} (1 - (y - 1)^{-1})$	$2g$
Negative entropy	$-\frac{1}{\log G} \sum_{g=1}^G \frac{N_g}{N} \log \frac{N_g}{N}$	$-\frac{y}{\log G}$	$\log \left(\frac{N_g}{N} \right)$

3 Polarization via additive inequality decomposition

In this section we consider multivariate inequality indices from the literature, which are additively decomposable.

Let us consider a multivariate inequality measure of type

$$I(\mathbf{X}) = f \left(\frac{1}{N} \sum_{i=1}^N h(s_i, \bar{s}) \right), \quad \mathbf{X} \in \mathcal{M}^{N \times K}. \quad (7)$$

Here $s_i = s_i(x_{i1}, \dots, x_{iK})$ signifies an individual evaluation function, $\bar{s}$ denotes a proper average either of the individual values s_i or of the attribute means $\bar{x}_k$, and f and h are continuous functions, f strictly increasing. We assume that, for some choice of f , h and s_i , $I(\mathbf{X})$ has an *additive decomposition by subgroups*,

$$I(\mathbf{X}) = B(\mathbf{X}) + W(\mathbf{X}) = B(\mathbf{X}) + \sum_{g=1}^G w_g I_g(\mathbf{X}), \quad (8)$$

where the inequality between groups and inside a group g are given, respectively, by

$$B(\mathbf{X}) = f \left(\sum_{g=1}^G \frac{N_g}{N} \cdot h(\bar{s}^g, \bar{s}) \right), \quad (9)$$

$$I_g(\mathbf{X}) = f \left(\frac{1}{N_g} \sum_{i \in g} h(s_i, \bar{s}^g) \right). \quad (10)$$

Here $\bar{s}^g$ is a mean like $\bar{s}$ that refers to group g , and w_g is a weight of group g .

From a multivariate inequality measure like this, polarization indices (2) to (4) are obtained. Table 2 lists five special decomposable measures that satisfy (7) to (10).

1. *Multivariate generalized entropy by Maasoumi.*

As an index of multivariate inequality, Maasoumi (1986) proposed the following generalized entropy measure (henceforth, GEM):

$$GEM_\gamma(\mathbf{X}) = \frac{1}{\gamma(1+\gamma)} \frac{1}{N} \sum_{i=1}^N \left[\left(\frac{s_i}{\bar{s}} \right)^{1+\gamma} - 1 \right], \quad \gamma \neq -1, 0, \quad (11)$$

$$GEM_{-1}(\mathbf{X}) = \frac{1}{N} \sum_{i=1}^N \log \left(\frac{\bar{s}}{s_i} \right), \quad (12)$$

$$GEM_0(\mathbf{X}) = \frac{1}{N} \sum_{i=1}^N \frac{s_i}{\bar{s}} \log \left(\frac{s_i}{\bar{s}} \right). \quad (13)$$

The attributes of each person, which have to be non-negative, i.e. $\mathbf{x}_i \in \mathbb{R}_+^K$, are aggregated through $s_i = (\sum_{k=1}^K \delta_k x_{ik}^{-\beta})^{-1/\beta}$, with $\delta_k \in [0, 1]$ and $\sum_{k=1}^K \delta_k = 1$. δ_k represents the weight of the k -th attribute and β is a constant that reflects the elasticity of substitution between attributes.

As proved in Maasoumi (1986), the GEM is additively decomposable in the sense of (8) to (10). The values of this index range from 0 to infinity; its components are shown in Table 2, where $\bar{s}$ is the arithmetic mean of the functions s_i over all N individuals, and $\bar{s}^g$ is the arithmetic mean of s_i over the individuals in subgroup g .

2. *Multivariate generalized entropy measure by Tsui.*

Another multivariate extension of the entropy measure (in the following, GET) has been introduced by Tsui (1999):

$$GET(\mathbf{X}) = \frac{\rho}{N} \sum_{i=1}^N \left(\prod_{k=1}^K \left(\frac{x_{ik}}{\bar{x}_k} \right)^{c_k} - 1 \right). \quad (14)$$

Such index imposes a restriction on the matrix $\mathbf{X}$: $x_{ik} > 0 \forall i, k$. The elements which constitute the GET measure are shown in Table 2. Here, the constants ρ and $c_1, \dots, c_K$ must satisfy particular conditions that are specified in Tsui (1999)³. This measure has its minimum at 0 and its supremum at infinity.

It is easily seen that, with group weights w_g given in Table 2, GET is an additively decomposable measure.

3. *Multivariate Kolm measure by Tsui.*

³In case of $K = 2$, the conditions on ρ and c_k are the following: $\rho c_1(c_1 - 1) > 0$, $c_1 c_2(1 - c_1 - c_2) > 0$, $\rho c_1 c_2 > 0$; they imply that $\rho > 0$, $c_1, c_2 < 0$.

A third multivariate measure which can be additively decomposed by subgroups is a generalization of Kolm's measure (in the following, KT), that has been introduced by Tsui (1995) and is given by:

$$KT(\mathbf{X}) = \frac{1}{\sum_{k=1}^K c_k} \ln \left\{ \frac{1}{N} \sum_{i=1}^N \exp \left\{ \sum_{k=1}^K c_k (\bar{x}_k - x_{ik}) \right\} \right\}. \quad (15)$$

However, the decomposition of KT differs slightly from that in the previous cases. It resembles the decomposition given by Blackorby et al. (1981) for the univariate Kolm index: the total inequality measure is the sum of the following within and between groups components:

$$\begin{aligned} W(\mathbf{X}) &= \sum_{g=1}^G \frac{N_g}{N} \left(\frac{1}{\sum_{k=1}^K c_k} \ln \left\{ \frac{1}{N_g} \sum_{i \in g} \exp \left\{ \sum_{k=1}^K c_k (\bar{x}_k^g - x_{ik}) \right\} \right\} \right), \\ B(\mathbf{X}) &= \frac{1}{\sum_{k=1}^K c_k} \ln \left\{ \sum_{g=1}^G \frac{N_g}{N} \exp \left\{ \sum_{k=1}^K c_k \left(\sum_g \frac{N_g}{N} \xi_g - \xi_g \right) \right\} \right\}, \end{aligned}$$

where

$$\xi_g = -\frac{1}{\sum_{k=1}^K c_k} \ln \left\{ \frac{1}{N_g} \sum_{i \in g} \exp \left\{ -\sum_{k=1}^K c_k x_{ik} \right\} \right\}$$

is an equivalent equally-distributed endowment of subgroup g , and c_k is a constant regarding the k -th attribute. For details, see Tsui (1995).

The total inequality I and the within groups inequality I_g have the form (7) and (10), respectively, with $s_i, \bar{s}$ and $\bar{s}^g$ shown in Table 2. The between component is, different from (9), not a function of $\bar{s}^g$ and $\bar{s}$, but of the ξ_g :

$$B^{KT}(\mathbf{X}) = f \left(\sum_{g=1}^G \frac{N_g}{N} \cdot h(\bar{\chi}^g, \bar{\chi}) \right), \quad (16)$$

with $\bar{\chi}^g = \sum_{k=1}^K c_k \xi_g$ and $\bar{\chi} = \sum_{k=1}^K c_k \sum_{g=1}^G \frac{N_g}{N} \xi_g$. The values of this measure range from 0 to $\frac{\sum_k c_k \bar{x}_k}{\sum_k c_k}$.

Note that the index KT allows also for negative values of the attributes, e.g. for negative wealth due to liabilities.

Table 2: Additively decomposable inequality measures $I(\mathbf{X})$ of type (7).

Index	$f(y)$	s_i	$\bar{s}$	$\bar{s}^g$	$h(t, \bar{t})$	w_g
$GEM_\gamma, \gamma \neq -1, 0$	$\frac{y}{\gamma(1+\gamma)}$	$\left(\sum_{k=1}^K \delta_k x_{ik}^{-\beta}\right)^{-\frac{1}{\beta}}$	$\frac{1}{N} \sum_{i=1}^N s_i$	$\frac{1}{N_g} \sum_{i \in g} s_i$	$(t/\bar{t})^{1+\gamma} - 1$	$\frac{N_g}{N} \left(\frac{\bar{s}^g}{\bar{s}}\right)^{1+\gamma}$
GEM_{-1}	y	$\left(\sum_{k=1}^K \delta_k x_{ik}^{-\beta}\right)^{-\frac{1}{\beta}}$	$\frac{1}{N} \sum_{i=1}^N s_i$	$\frac{1}{N_g} \sum_{i \in g} s_i$	$\log(\bar{t}/t)$	$\frac{N_g}{N}$
GEM_0	y	$\left(\sum_{k=1}^K \delta_k x_{ik}^{-\beta}\right)^{-\frac{1}{\beta}}$	$\frac{1}{N} \sum_{i=1}^N s_i$	$\frac{1}{N_g} \sum_{i \in g} s_i$	$(t/\bar{t}) \log(t/\bar{t})$	$\frac{N_g \bar{s}^g}{N \bar{s}}$
GET	$\rho \cdot y$	$\prod_{k=1}^K (x_{ik})^{c_k}$	$\prod_{k=1}^K (\bar{x}_k)^{c_k}$	$\prod_{k=1}^K (\bar{x}_k^g)^{c_k}$	$(t/\bar{t}) - 1$	$\frac{N_g}{N} \prod_{k=1}^K \left(\frac{\bar{x}_k^g}{\bar{x}_k}\right)^{c_k}$
KT	$\frac{\ln(y)}{\sum_k c_k}$	$\sum_{k=1}^K c_k x_{ik}$	$\sum_{k=1}^K c_k \bar{x}_k$	$\sum_{k=1}^K c_k \bar{x}_k^g$	$\exp\{\bar{t} - t\}$	$\frac{N_g}{N}$

4 Polarization via other inequality decompositions

The second group of multivariate indices of inequality here considered is given by the non-additively decomposable measures; in particular, we study multiplicatively decomposable indices and the multivariate Gini mean difference.

4.1 Multiplicative decomposition

Two special inequality measures that have a multiplicative decomposition are multivariate extensions of Atkinson's measure and have been introduced by Maasoumi (1986) (henceforth, AM) and Tsui (1995) (henceforth, AT). Tsui, in particular, proposes a double generalization, that will be indicated here with AT_1 and AT_2 .

$$AM_v(\mathbf{X}) = 1 - \left(\frac{1}{N} \sum_{i=1}^N \left(\frac{s_i}{\bar{s}} \right)^{1-v} \right)^{1/(1-v)}, \quad v > 0, \quad v \neq 1, \quad (17)$$

$$AM_1(\mathbf{X}) = 1 - \exp \left(\frac{1}{N} \sum_{i=1}^N \log \left(\frac{s_i}{\bar{s}} \right) \right). \quad (18)$$

$$AT_1(\mathbf{X}) = 1 - \left(\frac{1}{N} \sum_{i=1}^N \prod_{k=1}^K \left(\frac{x_{ik}}{\bar{x}_k} \right)^{r_k} \right)^{1/\sum_k r_k}, \quad (19)$$

$$AT_2(\mathbf{X}) = 1 - \exp \left(\frac{1}{N} \sum_{i=1}^N \log \left(\prod_{k=1}^K \left(\frac{x_{ik}}{\bar{x}_k} \right)^{r_k/\sum_j r_j} \right) \right). \quad (20)$$

Both assume values between 0 and 1 and have the form

$$I(\mathbf{X}) = 1 - A(\mathbf{X}),$$

where A is a multivariate similarity measure of the type

$$A(\mathbf{X}) = f \left(\frac{1}{N} \sum_{i=1}^N h(s_i, \bar{s}) \right), \quad (21)$$

with h , s_i and $\bar{s}$ as in (7) and f continuous and strictly monotone function.

For the particular functions f , h and s_i , chosen by Maasoumi (1986) and Tsui (1995), and following the approach of Lasso-de-la Vega and Urrutia (2003), the

similarity measure A in (21) can be multiplicatively decomposed into $A = A_B \cdot A_W$ or, equivalently,

$$\ln A = \ln A_B + \ln A_W,$$

where A_B and A_W are similarity measures, respectively, between and within groups, given by

$$A_B(\mathbf{X}) = f \left(\sum_{g=1}^G \frac{N_g}{N} \cdot h(\bar{s}^g, \bar{s}) \right), \quad (22)$$

$$A_W(\mathbf{X}) = \left(\sum_{g=1}^G w_g (A_g(\mathbf{X}))^\epsilon \right)^{1/\epsilon}, \text{ or} \quad (23)$$

$$A'_W(\mathbf{X}) = \left(\prod_{g=1}^G A_g(\mathbf{X}) \right)^{w_g}. \quad (24)$$

The first type of similarity measure within groups, A_W , is a weighted mean of order ϵ of the similarity measure inside each group, A_g , which is given by

$$A_g(\mathbf{X}) = f \left(\frac{1}{N_g} \sum_{i \in g} h(s_i, \bar{s}^g) \right).$$

This holds for measure AM_v with $v \neq 1$ and for the first measure of Tsui, AT_1 .

The second type of similarity within groups, A'_W in (24), holds for the measure AM_1 and for the second measure of Tsui, AT_2 .

Table 3 shows the particular components of the measures proposed both by Maa-soumi, with parameters $\delta_k \in [0, 1]$, $\sum_{k=1}^K \delta_k = 1$, and by Tsui, where the parameter r_k has to satisfy particular restrictions specified in Tsui (1995)⁴. Restrictions on matrix $\mathbf{X}$ are required by both the measures: $x_{ik} \geq 0$ for AM and $x_{ik} > 0$ for AT_1 and AT_2 , $\forall i, k$.

In case of multiplicative decomposition, we construct particular forms of (1) which are parallel to P_1, P_2, P_3 :

$$P_4(\mathbf{X}) = \phi \left(\frac{\ln A_B(\mathbf{X})}{\ln A_W(\mathbf{X}) + c} \right) \cdot S(\mathbf{X}); \quad (25)$$

$$P_5(\mathbf{X}) = \psi \left(\ln A_W(\mathbf{X}) - \ln A_B(\mathbf{X}) \right) \cdot S(\mathbf{X}); \quad (26)$$

$$P_6(\mathbf{X}) = \tau \left(\frac{\ln A(\mathbf{X}) + c}{\ln A_W(\mathbf{X}) + c} \right) \cdot S(\mathbf{X}), \quad (27)$$

with properly chosen ϕ, ψ and τ .

⁴For AT_1 , in case of $K = 2$, $r_1 \in (0, 1)$ and $r_2 < 1 - r_1$; for AT_2 , $r_k > 0$ for all $k = 1, \dots, K$.

Table 3: Multiplicatively decomposable similarity measures (21).

Index	$f(y)$	s_i	$\bar{s}$	$\bar{s}^g$	$h(t, \bar{t})$	w_g	ϵ
$AM_v, v \neq 1$	$y^{\frac{1}{(1-v)}}$	$\left(\sum_{k=1}^K \delta_k x_{ik}^{-\beta} \right)^{-\frac{1}{\beta}}$	$\frac{1}{N} \sum_{i=1}^N s_i$	$\frac{1}{N_g} \sum_{i \in g} s_i$	$(t/\bar{t})^{1-v}$	$\frac{\frac{N_g}{N} \left(\frac{\bar{s}^g}{\bar{s}} \right)^{1-v}}{\sum_{g=1}^G \frac{N_g}{N} \left(\frac{\bar{s}^g}{\bar{s}} \right)^{1-v}}$	$1 - v$
AM_1	$\exp(y)$	$\left(\sum_{k=1}^K \delta_k x_{ik}^{-\beta} \right)^{-\frac{1}{\beta}}$	$\frac{1}{N} \sum_{i=1}^N s_i$	$\frac{1}{N_g} \sum_{i \in g} s_i$	$\log(t/\bar{t})$	$\frac{N_g}{N}$	—
AT_1	$y^{\frac{1}{\sum_k r_k}}$	$\prod_{k=1}^K (x_{ik})^{r_k}$	$\prod_{k=1}^K (\bar{x}_k)^{r_k}$	$\prod_{k=1}^K (\bar{x}_k^g)^{r_k}$	$t/\bar{t}$	$\frac{\frac{N_g}{N} \prod_k \left(\frac{\bar{x}_k^g}{\bar{x}_k} \right)^{r_k}}{\sum_{g=1}^G \frac{N_g}{N} \prod_k \left(\frac{\bar{x}_k^g}{\bar{x}_k} \right)^{r_k}}$	$\sum_{k=1}^K r_k$
AT_2	$(\exp(y))^{\frac{1}{\sum_k r_k}}$	$\prod_{k=1}^K (x_{ik})^{r_k}$	$\prod_{k=1}^K (\bar{x}_k)^{r_k}$	$\prod_{k=1}^K (\bar{x}_k^g)^{r_k}$	$\log(t/\bar{t})$	$\frac{N_g}{N}$	—

4.2 Gini decomposition

The last inequality measure we consider here is a multivariate generalization of the Gini mean difference, the distance-Gini mean difference (see Koshevoy and Mosler (1997)), shortly GMD. It is given by

$$\Delta(\mathbf{X}) = \frac{1}{2KN^2} \sum_{i=1}^N \sum_{j=1}^N \|\mathbf{x}_i - \mathbf{x}_j\|,$$

where $\|\cdot\|$ indicates the Euclidean distance in $\mathbb{R}^K$. The distance-Gini mean difference is bounded between 0 and $\frac{1}{K} \sum_{k=1}^K \bar{x}_k$ and is defined also for negative endowments, $\mathbf{x}_i \in \mathbb{R}^K$; see Koshevoy and Mosler (1997).

To decompose the multivariate GMD, we follow the approach of Bhattacharya and Mahalanobis (1967) given for the univariate measure. By straightforward calculation we obtain

$$\begin{aligned} \Delta(\mathbf{X}) &= \sum_{g=1}^G \left(\frac{N_g}{N} \right)^2 \Delta_g + \frac{1}{2K} \sum_{g=1}^G \sum_{h \neq g}^G \frac{N_g N_h}{N^2} \|\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h\| \\ &+ \frac{1}{2K} \sum_{g=1}^G \sum_{h \neq g}^G \frac{N_g N_h}{N^2} \left\{ \sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} \|\mathbf{x}_i - \mathbf{x}_j\| - \|\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h\| \right\} \\ &= \Delta_W(\mathbf{X}) + \Delta_B(\mathbf{X}) + \Delta_{OV}(\mathbf{X}). \end{aligned}$$

In the previous equation, Δ_g , Δ_W and Δ_B represent the distance-Gini mean difference, respectively, inside group g , within all the groups and between groups. The residual component Δ_{OV} ,

$$\Delta_{OV}(\mathbf{X}) = \frac{1}{2K} \sum_{g=1}^G \sum_{h \neq g}^G \frac{N_g N_h}{N^2} \left\{ \sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} \|\mathbf{x}_i - \mathbf{x}_j\| - \|\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h\| \right\}, \quad (28)$$

corresponds to the univariate overlap component (see the following remarks) and is equal to zero if $\sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} \|\mathbf{x}_i - \mathbf{x}_j\| = \|\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h\|$ for every two groups g and h .

Remarks

- We choose the distance-Gini mean difference in place of the distance-Gini index since the distance-Gini index is difficult to decompose. However, there is no problem in using an absolute inequality index like the GMD: the essential consequence is that the polarization measures based on it is translation invariant instead of scale invariant; see Section 5.

- The multivariate overlap component Δ_{OV} is always non-negative, which is seen from

$$\Delta_{OV} = \sum_{g=1}^G \sum_{h \neq g}^G \frac{N_g N_h}{2KN^2} \left\{ \sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} \|\mathbf{x}_i - \mathbf{x}_j\| - \left\| \sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} (\mathbf{x}_i - \mathbf{x}_j) \right\| \right\}$$

and the triangle inequality.

- Necessary for $\Delta_{OV} = 0$ is that the groups have no ‘geometric overlap’, in the sense that their convex hulls do not intersect.
- With one attribute only, $\Delta_{OV} = 0$ if and only if there is no geometric overlap between the groups, that is, the groups are restricted to separate intervals. With more than one attribute the ‘if’ implication does not hold in general: Figure 1 shows an example with $N = 4$ and $K = G = 2$. The first group consists of endowment vectors $(1, 7)$ and $(3, 1)$, the second of $(4, 5)$ and $(6, 7)$. The two groups can be separated by a straight line, hence have no geometric overlap, but there holds $\Delta_{OV} = 0.157 > 0$.
- Sufficient for $\Delta_{OV} = 0$ is that there exists no inequality within groups. Then, all the individuals a group have endowment vector equal to the group mean; therefore:

$$\sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} \|\mathbf{x}_i - \mathbf{x}_j\| = \sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} \|\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h\| = \|\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h\|.$$

Another sufficient condition for $\Delta_{OV} = 0$ is that the endowment vectors $\mathbf{x}_i$ of all individual lie on a straight line, (i.e. the situation is essentially univariate) and, in addition, there is no geometric overlap among groups.

The polarization measures based on the distance-Gini mean difference are of the following types:

$$\begin{aligned} P_7(\mathbf{X}) &= \phi \left(\frac{\Delta_B(\mathbf{X})}{\Delta_W(\mathbf{X}) + \Delta_{OV}(\mathbf{X}) + c} \right) \cdot S(\mathbf{X}), \\ P_8(\mathbf{X}) &= \psi \left(\Delta_B(\mathbf{X}) - \Delta_W(\mathbf{X}) - \Delta_{OV}(\mathbf{X}) \right) \cdot S(\mathbf{X}), \\ P_9(\mathbf{X}) &= \tau \left(\frac{\Delta_B(\mathbf{X})}{\Delta(\mathbf{X}) + c} \right) \cdot S(\mathbf{X}), \end{aligned}$$

with functions ϕ, ψ and τ continuous and strictly increasing, properly chosen, and constant c positive.

Figure 1: Example: No geometric overlap (groups divided by a straight line), but $\Delta_{OV} > 0$.

5 Postulates for polarization indices

For univariate polarization measurement, a number of postulates or axioms have been presented in the literature. Part of them are continuity and invariance properties, others concern the minima and maxima of polarization indices and their monotonicity with respect to certain changes of the distribution. In this section we extend these postulates to the multivariate setting and discuss whether they are satisfied with the multivariate indices introduced in Sections 3 and 4.

5.1 Invariance and continuity postulates

A first group of postulates concerns the continuity of a multivariate polarization index P and its invariance with respect to certain transformations of the matrix $\mathbf{X}$. In our setting, such properties are generally inherited from the same properties of the indices B , W , S , and I on which P is based.

1. **Continuity** P is continuous as a function of $\mathbf{X} \in \mathcal{M}^{N \times K}$.
2. **Anonymity** P is invariant to the individual labels. Formally, for any $N \times N$ permutation matrix Π , the postulate requires that $P(\mathbf{X}) = P(\Pi\mathbf{X})$.
3. **Replication Invariance** The index depends on the frequency distribution of endowments only. Formally, let $\mathbf{Y}$ be the matrix obtained by repeating $\mathbf{X}$ matrix H times, such that the number of columns of $\mathbf{Y}$ is K and the number of rows is $N \times H$. The property requires that $P(\mathbf{Y}) = P(\mathbf{X})$. It means that replicating the population, without changing the distribution of the variables, does not influence polarization.

4. **Weak Scale Invariance** The index does not depend on a common scale factor. Formally, $P(\lambda \mathbf{X}) = P(\mathbf{X})$ for all $\lambda > 0$.
5. **Strong Scale Invariance** The index does not depend on the units of measurement of the attributes, it is a *relative index*. Formally, $P(\mathbf{X}\Lambda) = P(\mathbf{X})$ if $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_K)$, with $\lambda_i > 0, i = 1, \dots, K$.
6. **Translation Invariance** The index does not change when each individual receives the same additional vector of endowments. Formally, $P(\mathbf{X} + \Lambda) = P(\mathbf{X})$ if Λ is an $N \times K$ matrix with all identical rows. A translation invariant index is also called an *absolute index*.

For each of these postulates holds in general: a multivariate polarization index P of type (1) satisfies the postulate if the indices B, W, S and I on which it is based do. By this, all special indices introduced in Section 3 are continuous and satisfy Anonymity and Replication Invariance.

Obviously, the size indices (5) and (6) of Subsection 2.3 are scale and translation invariant. Moreover, among the multivariate inequality measures considered, GEM, GET, AM and AT are scale invariant⁵. Therefore, also the corresponding polarization measures P are scale invariant. But, the polarization indices constructed from the distance-Gini mean difference (GMD) and the KT index satisfy neither Strong nor Weak Scale Invariance, as the underlying inequality measures are no relative ones. On the other hand, GMD and KT are absolute indices; they satisfy Translation Invariance. Therefore, the polarization indices constructed from them are absolute, too.

5.2 Polarization properties

A second group of properties pertains properly to the polarization concept, i.e. to the double tendency of the groups to be internally homogeneous and externally heterogeneous.

1. **Maximum Polarization** In univariate polarization measurement (Esteban and Ray (1994), Wolfson (1997), Milanovic (2000)), the following situation is regarded as the extreme case in which the society is perfectly polarized: the society divides into two groups of identical size (the rich and the poor), and the groups are completely homogeneous inside (i.e. without any internal inequality) and at maximum distance to each other, given the income endowment of the entire society. It means that polarization is maximum when population is made up of individuals, half of which have all the same minimum level of

⁵GEM fulfils the weak version only, while the others also satisfy the strong version.

income and the others have all the same maximum income. Esteban and Ray (1994) consider such minimum and maximum levels of income as, respectively, the lowest and the highest of the income classes into which the population is a priori partitioned; Milanovic (2000) and Wolfson (1997) sharpen this postulate in saying that the upper level of income is equal twice the mean, and the lower level is 0.

Analogously, in the multivariate context, we postulate that a two-groups society shows **maximum polarization** if it consists of two equally large groups, the individuals in each group have the same endowment vector and the mean vectors of the two groups are at maximum distance. More generally, given a number G of groups, polarization is considered maximum if the groups are equally sized, internally homogeneous and the group mean vectors show maximum disparity as measured by a proper inequality index.

It is obvious from the formula (1) and the assumptions in Subsection 2.1 that every polarization measure P satisfies the Maximum Polarization property.

2. **Minimum Polarization** The ‘normalization axiom’ of univariate polarization measurement (Wang and Tsui (2000), Chakravarty and Majumder (2001)) states that polarization reaches its minimum value ($= 0$) when all the individuals have the same income, i.e. in the case of an egalitarian distribution.

Gradín (2000) postulates, instead, that polarization is minimized if there is both perfect equality between groups and maximum intra-group disparity; in particular, minimum polarization arises if the groups which constitute the population have null intergroup inequality and, inside each groups, inequality is maximum. For our indices P it is obvious from the formula (1) that **minimum polarization** is obtained when B and S are minimized and W is maximized, that is, when the population is constituted by only one group and inequality is maximum. Hence, P satisfies Gradín’s postulate, but not the above ‘normalization axiom’.

3. **Increased Spread** The ‘increased spread’ property of univariate polarization measurement (Wang and Tsui (2000), Chakravarty and Majumder (2001)) establishes that, given two groups, if any individual of one group moves further from the other group, polarization increases.

To extend this notion to the multivariate case, we consider shifts of two or more groups that increase the dispersion of their group means. A group g is shifted by some $\mathbf{c}^g \in \mathbb{R}^K$ if the endowment vector X_i of each member $i \in g$ is shifted to $Y_i = X_i + \mathbf{c}^g$. Consequently the mean $\bar{\mathbf{x}}^g$ of group g is shifted to $\bar{\mathbf{y}}^g = \bar{\mathbf{x}}^g + \mathbf{c}^g$. To describe increasing dispersion of group means, we employ four different notions of multivariate majorization.

Consider matrices $\mathbf{U}$ and $\mathbf{V}$ that have format $M \times K$. Each of the following

six notions reduces to univariate Pigou-Dalton majorization⁶ when $K = 1$:

- (a) $\mathbf{U} \succ_T \mathbf{V}$ if $\mathbf{U} = A\mathbf{V}$ with A = finite products of T - matrices, where $T = \lambda I + (1-\lambda)Q$, $\lambda \in [0, 1]$, I is the identity matrix, and Q a permutation matrix that interchanges only two coordinates;
- (b) $\mathbf{U} \succ_B \mathbf{V}$ if $\mathbf{U} = B\mathbf{V}$ where B is an $M \times M$ bistochastic matrix;
- (b') $\mathbf{U} \succ_c \mathbf{V}$ if $\mathbf{U} \in \text{conv}\{\Pi\mathbf{V} : \Pi \text{ is a } M \times M \text{ permutation matrix}\}$;
- (c) $\mathbf{U} \succ_p \mathbf{V}$ if $\mathbf{U}\mathbf{p}^T = B_k \cdot \mathbf{V}\mathbf{p}^T$, $\mathbf{p} \in \mathbb{R}^K$ and $B_k = M \times M$ bistochastic matrix specific for $k = 1, \dots, K$;
- (c') $\mathbf{U} \succ_k \mathbf{V}$ if $\mathbf{u}^k = B_k \mathbf{v}^k$, with $\mathbf{v}^k = k$ -th column of $\mathbf{V}$ and $B_k = M \times M$ bistochastic matrix, $\forall k = 1, \dots, K$;
- (d) $\mathbf{U} \succ_L \mathbf{V}$ if $LZ(\mathbf{U}) \subset LZ(\mathbf{V})$, with $LZ(\mathbf{U}) =$ Lorenz zonoid of distribution $\mathbf{U}$.

However, the six notions are not equivalent; in fact: (a) $\Rightarrow$ (b) $\Leftrightarrow$ (b') $\Rightarrow$ (c) $\Leftrightarrow$ (c') $\Rightarrow$ (d). For details, see Mosler (1994) and Marshall and Olkin (1979).

We propose the following **multidimensional increased spread property**: whenever two or more groups are shifted such that their means become more dispersed in terms of majorization (a), (b), (c) or (d), then polarization increases.

Neither the inequality W within groups nor the groups size measure S are modified by a majorization movement of the group centers; the only component of the measure P that is involved is the inequality between groups B . That is, the polarization measure (1) satisfies the multidimensional increased spread property (a), (b), (c) or (d) if and only if the between groups inequality measure increases under majorization (a), (b), (c) or (d), respectively.

Every multivariate inequality measure used in Sections 3 and 4 increases with one of these majorizations. In particular, the measures GEM, GET, KT, AM, and AT satisfy the property with (b), while GMD is increasing with (d) (and the majorizations that imply these). Therefore, all polarization measures obtained from these inequality indices fulfil the property.

4. **Increased Polarity** The univariate version of this property (often called 'increased bipolarity'; see Wang and Tsui (2000), Chakravarty and Majumder (2001)) requires that a Pigou-Dalton transfer within one or more groups increases polarization. It means that if, inside a group, one distribution is obtained from the other by univariate Pigou-Dalton majorization, then the polarization in the first distribution is higher than in the second.

In the multidimensional case, we say that the **increasing polarity** property of type (a), (b), (c) or (d) holds if polarization increases whenever the population

⁶that is, $\mathbf{U} \succ_{PD} \mathbf{V}$ if $\mathbf{U}$ is obtained from $\mathbf{V}$ by a finite sequence of Pigou-Dalton transfers.

in one of the groups is exchanged against a majorizing population of type (a), (b), (c) or (d), respectively. For the polarization measure (1) the property is satisfied if and only if the within inequality W decreases with a majorization (a), (b), (c) or (d).

Obviously, each multivariate inequality measures considered in Sections 3 and 4 satisfies one of these notions. In particular, the measures GEM, GET, KT, AM, AT respect majorization (b), while GMD majorization (d) (and as well the majorizations that imply these). By this, all polarization measures obtained from these inequality indices fulfil the increased polarity property in one of the four versions.

6 Correlation and interaction among attributes

Correlation increasing majorizations The last type of properties we consider is peculiar to multivariate analysis, as it takes into account the interaction between the different variables involved in the analysis. In particular, we study the effect on the polarization measure of transfers that increase the correlation between the attributes. We define, at first, a **correlation increasing transfer** similar to that in Tsui (1999).

Definition 6.1. *Matrix $\mathbf{Y}$ is obtained from $\mathbf{X}$, with $\mathbf{X}, \mathbf{Y} \in \mathcal{M}^{N \times K}$, by a correlation increasing transfer (CIT) if $\mathbf{X} \neq \mathbf{Y}$, $\mathbf{X}$ is not a permutation of $\mathbf{Y}$ and there exist $i, j \in \{1, 2, \dots, N\}$ such that $\mathbf{y}_i = \mathbf{x}_i \wedge \mathbf{x}_j$, $\mathbf{y}_j = \mathbf{x}_i \vee \mathbf{x}_j$ and $\mathbf{y}_h = \mathbf{x}_h$ for all $h \notin \{i, j\}$.*

Here, $\mathbf{x} \wedge \mathbf{y} = \{\min(x_1, y_1), \dots, \min(x_K, y_K)\}$ and $\mathbf{x} \vee \mathbf{y} = \{\max(x_1, y_1), \dots, \max(x_K, y_K)\}$ denote the componentwise maximum and minimum of $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^K$.

For a multivariate inequality measure I , the *correlation increasing majorization* (Weymark (2006)) establishes that if $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a CIT, then, if the attributes are considered as substitutes, $I(\mathbf{Y}) \geq I(\mathbf{X})$, while, if the attributes are considered as complements, $I(\mathbf{Y}) \leq I(\mathbf{X})$.

In our multivariate polarization measures, however, two opposite directions are present, B and W . Therefore, it seems more natural to study the behavior of the measure P with respect to between and within groups correlation increasing transfers separately.

A *between groups correlation increasing transfer (BCIT)* is a transfer that increases the correlation between the attributes only in reference to the means of the groups, i.e. a CIT in which the individual incomes are replaced by the groups' means.

Now we are able to define the relevant property of a multivariate polarization measure P . P satisfies the **between groups correlation increasing majorization**

property if $P(\mathbf{Y}) \geq (\leq) P(\mathbf{X})$ whenever $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a *BCIT* and the attributes are substitutes (complements).

Consider now a *within groups correlation increasing transfer (WCIT)*, i.e. a transfer which increases the correlation between the attributes only for individuals inside a given group.

P satisfies the **within groups correlation increasing majorization** property if $P(\mathbf{Y}) \leq (\geq) P(\mathbf{X})$ whenever $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a *WCIT* and the attributes are substitutes (complements).

Among the inequality measures considered in Sections 3 and 4, the only index which satisfies the correlation increasing majorization is GET. Such measure considers, however, the attributes only as perfect substitutes. In presence of correlation increasing transfers, in fact, GET can only increase.

Therefore, the polarization measures $P_i, i = 1, 2, 3$ obtained from GET increase, in presence of *BCIT*, and decrease, in case of *WCIT*.

Interaction between attributes We further have to take into account what kind of interaction among the variables is evaluated by the researcher (or by society). Multivariate inequality increases when the variability of an attribute increases. It also increases when the correlation between variables rises and the variables are substitutes; it decreases when they are complements. Consequently, the results of polarization measurement are different.

The importance of considering the interaction between the attributes has been underlined in Maasoumi and Nickelsburg (1988) and in Bourguignon and Chakravarty (2003); in these papers, an appropriate parameter is introduced, which reflects the evaluation of the researcher, or of the society, on the relationship between the variables.

We have, therefore, to choose measures of inequality that allow the polarization measures to express our evaluation on the interactions between the attributes.

Some of the inequality measures considered above are so flexible to allow for different kinds of association between attributes; they are the GEM and AM measures. The aggregative function s_i , introduced by Maasoumi (1986), is, in fact, based on the parameter β , which expresses the degree of substitution between attributes, such that $\beta = (1/\sigma) - 1$, where σ is a constant elasticity of substitution. So, if two attributes are substitutes, σ tends to infinity and, correspondingly, $\beta \rightarrow -1$. If they are complements, $\sigma \rightarrow 0$ and $\beta \rightarrow \infty$. $\sigma = 1$ and $\beta = 0$ means an intermediate situation with a certain degree of substitution between attributes.

The other inequality measures of Sections 3 and 4 do not possess such flexibility: the GET measure regards all goods as substitutes, ignoring the case of complements; the measures AT, KT and GMD, instead, do not consider this aspect of evaluation.

7 Numerical illustration

In order to illustrate the proposed measures, we now present a simple numerical example.

Consider the joint distribution of income and wealth over a population of 10 individuals, in two different periods of time. The population is split into two groups of equal size; therefore the groups size measure S is maximum ($= 1$).

The multivariate distributions for the first and the second period of time are given, respectively, by the following matrices $\mathbf{X}_1$ and $\mathbf{X}_2$, where the first column represents income and the second wealth:

$$\mathbf{X}_1 = \begin{bmatrix} 10 & 5 \\ 20 & 10 \\ 25 & 20 \\ 15 & 20 \\ 20 & 35 \\ 50 & 40 \\ 40 & 55 \\ 50 & 55 \\ 75 & 40 \\ 65 & 75 \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} 15 & 10 \\ 20 & 10 \\ 15 & 20 \\ 25 & 20 \\ 20 & 30 \\ 60 & 50 \\ 50 & 60 \\ 60 & 65 \\ 65 & 55 \\ 70 & 70 \end{bmatrix}.$$

Figure 2 shows the plots of the two distributions. A first visual inspection points at an increase in polarization when moving from $\mathbf{X}_1$ to $\mathbf{X}_2$.

In order to include in the analysis the degree of substitution between income and wealth, we are induced to restrict the class of polarization measures only to the ones obtained from the inequality indices GEM and AM, as discussed in Section 6. As the rankings of the distributions obtained from the two indices are ordinally equivalent, we focus here on the GEM measure and we calculate the polarization indices P_1^{GEM} , P_2^{GEM} and P_3^{GEM} obtained from it, described in Subsection 2.2.

To keep the analysis at a general level, we allow the two attributes to be substitutes or complements, according to the researcher's evaluation. For the complementary case, β is set equal to 5, for perfect substitution $\beta = -1$, and for an intermediate case $\beta = -0, 5$. The results are shown in Table 4.

From the Table 4, we observe that polarization increases, according to every value of parameter β , while multivariate inequality decreases. This shows that, as in the univariate case, polarization and inequality constitute two distinct aspects of a distribution.

Notice that the three measures P_1^{GEM} , P_2^{GEM} and P_3^{GEM} order the two distributions in the same way. Moreover, the less substitutes are considered the attributes

Figure 2: Plots of distributions $\mathbf{X}_1$ and $\mathbf{X}_2$.

Table 4: Inequality and polarization of the distributions $\mathbf{X}_1$ and $\mathbf{X}_2$.

	GEM		P_1^{GEM}		P_2^{GEM}		P_3^{GEM}	
	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_2$
$\beta = -1^*$	0.1507	0.1490	0.1157	0.1300	0.53214	0.54174	0.1095	0.1222
$\beta = -0.5^*$	0.1524	0.1510	0.1169	0.1315	0.53251	0.54223	0.1106	0.1235
$\beta = 5^*$	0.1723	0.1684	0.1266	0.1445	0.53402	0.54649	0.1192	0.1350
$d = (0.8; 0.2)^{**}$	0.1567	0.1467	0.1180	0.1282	0.53220	0.54116	0.1116	0.1207
$d = (0.2; 0.8)^{**}$	0.1611	0.1548	0.1117	0.1314	0.52693	0.54128	0.1059	0.1235

* $g = 1.5$; $d = (0.5; 0.5)$; $c = 1$.

** $g = 1.5$; $\beta = -1$; $c = 1$.

(i.e. when the parameter β increases), the more polarized and the more unequal is regarded the society.

The last two lines of Table 4 show how polarization values change, weighting in a different way the two attributes. We can observe that the increment of polarization, moving from distribution $\mathbf{X}_1$ to $\mathbf{X}_2$, is higher if wealth is more weighted than income⁷; the opposite is true for the inequality measure.

An analogous characteristic can be seen in the univariate distributions; Table 5

⁷Comparing the distribution in the two different periods, the measure P_1 registers an increment of 8%, in case of $d = (0.8; 0.2)$ and of 15% for $d = (0.2; 0.8)$; P_2 gives 17% in the first case and 27% in the second; with P_3 we have, respectively, 8% and 14%.

shows the values of inequality (Gini index) and polarization (Tsui index; see Wang and Tsui (2000)) of income and wealth in the two different periods of time.

Table 5: Univariate inequality and polarization measures of income and wealth.

	Index	Period 1	Period 2
Income	Gini	0.3216	0.2975
	Tsui	0.7378	0.7407
Wealth	Gini	0.3338	0.3205
	Tsui	0.6294	0.7122

Polarization increases both in income and in wealth distribution, pointing out a trend, in this particular example, analogous to the trend of multivariate polarization. Table 5 shows also that the level of polarization is higher in income than in wealth and that the increase in univariate polarization, moving from the period 1 to the period 2, is more pronounced in wealth than in income distribution⁸; this is reflected in the higher increment in multivariate polarization that we have noticed, weighting wealth more than income.

8 Concluding remarks

We have proposed a multidimensional approach to polarization measurement, in order to include monetary and non-monetary attributes besides income. Our point of view on polarization focuses on the presence of two or more groups in the society, which are similar inside, distant to each other and equal in size.

We have proposed a new class of multivariate polarization indices, which are functions of three components: inequality between groups, inequality within groups and groups size. We have introduced indices of groups size, which measure the degree of similarity in population shares among the clusters. Exploiting the decomposition by groups of certain multivariate inequality measures, we have then used the two components of between and within inequality, in order to obtain a general class of multivariate polarization measures.

The new indices are general, in that they apply to any grouped distribution and require neither fixed groups nor fixed relative groups sizes. They evaluate the total data and their grouping as well; moreover, they may also be used to compare alternative groupings.

Many properties have been investigated, which are multidimensional extensions of the classical univariate polarization axioms, and the conditions, under which our indices satisfy them, have been analyzed.

⁸Income polarization increases by 0.3% and wealth polarization by 11.6%.

In the multi-attribute analysis, interactions between attributes have to be taken into account. We have handled this problem from an evaluative point of view, considering their association in terms of substitutional or complementary goods. If one ignores such aspect, all above multivariate inequality measures can be used to construct a polarization measure of form (1). However, if interactions are considered as relevant, the range of choices in our approach is reduced to those inequality measures which allow for such evaluation; in particular, the multivariate extensions of the generalized entropy measure and of the Atkinson's index proposed by Maasoumi (1986) are appropriate for such intent.

From an empirical point of view, finally, it seems interesting to apply the polarization measures proposed here to microdata, in order to analyze the trend of polarization in a multi-attribute context, taking into account other variables beyond income, such as wealth, education and health.

References

- ATKINSON, A. and BOURGUIGNON, F. (1982). The comparison of multidimensional distributions of economic status. *The Review of Economic Studies* **49**, 183–201.
- BHATTACHARYA, N. and MAHALANOBIS, B. (1967). Regional disparities in household consumption in India. *Journal of the American Statistical Association* **62**, 143–161.
- BLACKORBY, C., DONALDSON, D. and AUERSPERG, M. (1981). A new procedure for the measurement of inequality within and among population subgroups. *Canadian Journal of Economics* **14**, 665–685.
- BOURGUIGNON, F. and CHAKRAVARTY, S. R. (2003). The measurement of multidimensional poverty. *Journal of Economic Inequality* **1**, 25–49.
- CHAKRAVARTY, S. R. and MAJUMDER, A. (2001). Inequality, polarization and welfare: Theory and applications. *Australian Economic Papers* **40**, 1–13.
- D'AMBROSIO, C. (2001). Household characteristics and the distribution of income in Italy: An application of a social distance measures. *Review of Income and Wealth* **47**, 43–64.
- DAVIS, J. and HUSTON, J. (1992). The shrinking middle-income class: A multivariate analysis. *Eastern Economic Journal* **18**, 277–285.
- DUCLOS, J., ESTEBAN, J. and RAY, D. (2004). Polarization: Concepts, measurement, estimation. *Econometrica* **72**, 1737–1772.

- ESTEBAN, J., GRADÍN, C. and RAY, D. (1999). Extension of a measure of polarization with an application to the income distribution of five OECD countries. Working Paper Series, 218, Luxembourg Income Study, Differdange, Luxembourg.
- ESTEBAN, J. and RAY, D. (1994). On the measurement of polarization. *Econometrica* **62**, 819–851.
- GRADÍN, C. (2000). Polarization by sub-populations in Spain, 1973-91. *Review of Income and Wealth* **46**, 457–474.
- KOLM, S. (1977). Multidimensional egalitarianisms. *The Quarterly Journal of Economics* **91**, 1–13.
- KOSHEVOY, G. and MOSLER, K. (1997). Multivariate Gini indices. *Journal of Multivariate Analysis* **60**, 252–276.
- LAZZO-DE-LA VEGA, M. and URRUTIA, A. (2003). A new factorial decomposition for the Atkinson measure. *Economics Bulletin* **4**, 1–12.
- MAASOUMI, E. (1986). The measurement and decomposition of multidimensional inequality. *Econometrica* **54**, 991–997.
- MAASOUMI, E. and NICKELSBURG, G. (1988). Multivariate measures of well-being and an analysis of inequality in the Michigan data. *Journal of Business and Economic Statistics* **6**, 327–334.
- MARSHALL, A. and OLKIN, I. (1979). *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York.
- MILANOVIC, B. (2000). A new polarization measure and some applications. Mimeo, Development Research Group, World Bank.
- MOSLER, K. (1994). Majorization in economic disparity measures. *Linear Algebra and Its Applications* **199**, 91–114.
- PIESCH, W. (1975). *Statistische Konzentrationsmasse*. J.C. Mohr, Tuebingen.
- RODRÍGUEZ, J. and SALAS, R. (2003). Extended bi-polarization and inequality measures. *Research on Economic Inequality* **9**, 69–83.
- TSUI, K. (1995). Multidimensional generalizations of the relative and absolute inequality indices: The Atkinson-Kolm-Sen approach. *Journal of Economic Theory* **67**, 251–265.
- TSUI, K. (1999). Multidimensional inequality and multidimensional generalized entropy measures: An axiomatic derivation. *Social Choice and Welfare* **16**.

- WANG, Y. and TSUI, K. (2000). Polarization orderings and new classes of polarization indices. *Journal of Public Economic Theory* **2**, 349–363.
- WEYMARK, J. (2006). The normative approach to the measurement of multidimensional inequality. In F. Farina and E. Savaglio, eds., *Inequality and Economic Integration*, 303–328. Routledge, London.
- WOLFSON, M. C. (1994). When inequalities diverge. *The American Economic Review* **48**, 353–358.
- WOLFSON, M. C. (1997). Divergent inequalities: Theory and empirical results. *Review of Income and Wealth* **43**, 401–421.
- ZHANG, X. and KANBUR, R. (2001). What difference do polarisation measures make? An application to China. *The Journal of Development Studies* **37**, 85–98.