

# Single-Crossing, Strategic Voting and the Median Choice Rule\*

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## Abstract

This paper studies the strategic foundation of the Representative Voter Theorem (Rothstein, 1991), also called the “second version” of the Median Voter Theorem. As a by-product, it also considers the existence of non-trivial strategy-proof social choice functions over the domain of single-crossing preferences. The main result shows that single-crossing constitutes a domain restriction over the real line that allows not only majority voting equilibria, but also non-manipulable choice rules. In particular, this is true for the median choice rule, which is found to be strategy-proof and group-strategic-proof over the full set of alternatives and over every possible policy *agenda*. In addition, the paper also examines the close relation between single-crossing and order-restriction. And it uses this relation together with the strategy-proofness of the median choice rule to prove that the collective outcome predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism. This mechanism is a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the winner chooses a policy to be implemented by the

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## 1 Introduction

In the last twenty five years, *single-crossing* has become a “popular” feature of preferences within the field of Political Economy.<sup>1</sup> From the seminal works of Roberts (1977) and Grandmont (1978) and, more recently, due to the theoretical developments of Rothstein (1990, 1991), Gans and Smart (1996) and Austen-Smith and Banks (1999), it is now well-known that this domain restriction is sufficient to guarantee the existence of equilibria in one-dimensional models of majority voting, especially in situations where single-peakedness may not hold.

Moreover, this restriction is not only technically convenient, but it also makes sense in many political settings. In few words, the single-crossing property used in the context of voting, which is similar to that used in the principal-agent literature and for monotone comparative statics, says that, given any two policies, one of them more to the right than the other, the more rightist is an individual (with respect to another individual) the more he will “tend to prefer” the right-wing policy over the left-wing one. (The exact meaning of “tend to prefer” is made clear in the next section, in Definition 2.)

Thus, unlike single-peakedness, single-crossing is a restriction that imposes limitations *across* individual preferences, on the character of the voters’ heterogeneity, rather than on the shape of individual preferences. The main idea behind it is that, in many circumstances, *ordering* people according to a single parameter (like income, productivity, intertemporal preferences, ideological position, etc.) may be more natural than ordering alternatives. Under this condition, the conflict of interests among individuals is projected into a one-dimensional parameter space, and then the *types* of the agents are assigned a position over this left-right scale with the requirement that, for any pair of alternatives, the set of types preferring one of the alternatives all lie to one side of those who prefer the other.

It turns out that this condition not only guarantees the existence of majority voting equilibria, but it also provides a simple characterization of the core of the majority rule. In fact, the core is simply the set of ideal points of the median type *agent* in the ordering of individuals with respect to which

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<sup>1</sup>See, for example, the different applications found in Persson and Tabellini (2000).

the preference profile is single-crossing.<sup>2</sup> This result is sometimes referred to in the literature as the *Representative Voter Theorem* (Rothstein, 1991) (henceforth RVT) or, alternatively, as “the second version” of the Median Voter Theorem (Myerson, 1996 and Gans and Smart, 1996).

The main problem with this result is that, unlike the *original* Median Voter Theorem over single-peaked preferences, whose non-cooperative foundation was provided by Black (1948), first, and then by Moulin (1980), the RVT is based on the assumption that individuals honestly reveal their preferences. That is, it is derived assuming *sincere voting*. Clearly, this assumption is difficult to maintain in applications that focus on policy choices made in strategic frameworks. Hence, a natural question arises with respect to its applicability in those models.

This paper studies the strategic foundation of the Representative Voter Theorem. As a by-product, it also considers the existence of non-trivial strategy-proof social choice functions on the domain of single-crossing preferences and over the real line. There are several reasons that justify carrying out this analysis. But the first and more important one is that, even though single-crossing is now largely used in models of collective decision-making, nothing has been said in the literature about the possibility of manipulation over this domain. In particular, the “single-crossing version” of the Median Voter Theorem is used without caring much about its strategic foundation. So, one of the main purposes here is filling out this gap.

In addition, the study is also motivated by a more technical fact, though not less important. The analysis of strategic voting in the context of single-crossing preferences leads to consider strategy-proofness over a preference domain where there exists a specific kind of *interdependence* among individual preferences. This contrasts with much of the work developed in the field, which focuses on social choice rules defined over Cartesian preference domains. Moreover, this feature looks interesting for studying manipulation in multi-dimensional choice spaces and over constrained sets of alternatives, a problem that is extremely important in Political Economy (where voters usually have to choose from sets with only a few policies, rather than from the full set of alternatives).

The main result of the paper shows that single-crossing constitutes a domain restriction over the real line that allows not only majority voting equilibria, but also non-manipulable choice rules. In particular, this is true for the median choice rule, which is found to be strategy-proof and group-strategic-

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<sup>2</sup>In contrast, under single-peaked preferences, the core of the majority rule consists of the median *ideal points* in the ordering of alternatives with respect to which the profile is single-peaked.

proof over the full set of alternatives and over every possible policy *agenda*. The paper also shows the close relation between single-crossing and order-restriction; and it extends the former (by introducing the notion of *broad single-crossing*) to make both concepts equivalent. Finally, it uses this relation together with the strategy-proofness of the median choice rule to prove that the collective outcome predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism. This mechanism is a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

The paper is organized as follows. Section 2 presents the model, the notation and the definitions. Section 3 exhibits the equivalence between single-crossing and order-restriction for preferences indexed by the types of the agents. Section 4 presents the non-strategic version of the Representative Voter Theorem (the “order-restricted version” of the Median Voter Theorem). The results related to strategy-proofness and the *indirect* implementation of the median choice rule over single-crossing preferences are presented in section 5. In this section, the results of section 3 are also used to derive the game-theoretic counterpart of the Representative Voter Theorem. Final remarks and extensions are discussed in section 6.

## 2 The model, notation and definitions

Consider a society with a finite number of agents, represented by the elements of the set  $I = \{1, \dots, n\}$ , where  $|I| = n$  is odd and  $n > 2$ . These agents face a collective choice problem, which consists in choosing an alternative (for example, the level of a public good) from a finite subset of the real line. They make this choice by voting.

The set of all possible outcomes is  $X = \{x_1, \dots, x_l\}$ ,  $|X| > 2$ , where  $X$  is a finite subset of the non-negative real line  $\mathfrak{R}_+$ . The set of feasible alternatives may be either the entire  $X$  or just one of its non-empty subsets. We denote  $\tilde{X}$  a generic subset of  $X$  and  $A(X) = \{\tilde{X} : \tilde{X} \in 2^X \setminus \emptyset\}$ . In words,  $X$  is the universal set of outcomes, whereas a particular situation, or *agenda*, involves a  $\tilde{X} \in A(X)$ . Following the standard notation, for a vector  $(x_1, \dots, x_n) \in \mathfrak{R}_+^n$ , we let  $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  and  $(\hat{x}_i, x_{-i}) = (x_1, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_n)$ , where  $\hat{x}_i \in \mathfrak{R}_+$ . In addition, for any group of agents  $S \subseteq I$ , we denote  $(x_S, x_{\bar{S}}) = ((x_i)_{i \in S}, (x_j)_{j \in \bar{S}})$ , where  $\bar{S} = I \setminus S$ .

Let  $P(X)$  be the set of all complete, transitive and antisymmetric binary orderings of  $X$ . We say  $P(X)$  is the *universal domain* of individual prefer-

ences.<sup>3</sup> We assume agent  $i$ 's preferences over  $X$  are completely characterized by a single parameter  $\theta_i \in \Theta \subset \mathfrak{R}$ . As usual, we interpret  $\theta_i$  as being agent  $i$ 's *type*. That is, we assume there exists a function  $\Phi : \Theta \rightarrow P(X)$  that assigns a unique binary relation  $\Phi(\theta) \in P(X)$  to each type  $\theta \in \Theta$ . Then, we say that  $\succ_i$  represents the preferences of an agent  $i$  of type  $\theta_i$  if,

$$\forall x, y \in X, x \succ_i y \Leftrightarrow x \Phi(\theta_i) y.$$

The following example illustrates how these preferences can arise naturally in political-economic models:

**Example 1 (Persson and Tabellini, 2000)** *Consider the following simplified version of the redistributive distortionary taxation model of Roberts (1977). Suppose individual  $i \in I$  has preferences  $u(c_i, l_i) = c_i + v(l_i)$ , where  $c_i$  denotes individual consumption,  $l_i$  leisure and  $v_l > 0$  and  $v_{ll} \leq 0$ .<sup>4</sup> The individual's budget constraint is  $c_i \leq (1 - t)h_i + f$ , where  $t \in (0, 1)$  is an income tax rate,  $f \in \mathfrak{R}_+$  a lump-sum transfer and  $h_i$  the individual labor supply. The real wage is exogenous and normalized at unity. Individuals are heterogenous in a productivity parameter  $\theta_i \in \Theta \subset \mathfrak{R}$ , which is distributed in the population with mean  $\bar{\theta}$ . Given these different productivities, each individual  $i$  faces an "effective" time constraint  $1 - \theta_i \geq l_i + h_i$ . Finally, the government runs a balanced budget, so that  $nf \leq t \sum_{i \in I} h_i$ .*

*Solving the model and substituting into the individual utility function, the induced preferences of  $i$  over the different tax rates can be expressed as*

$$w_i(t) = h(t) + v[1 - h(t) - \bar{\theta}] - (1 - t)(\theta_i - \bar{\theta}),$$

*where  $h(t) = 1 - \bar{\theta} - v_l^{-1}(1 - t)$  is the average labor supply. Clearly, for each individual  $i \in I$ ,  $w_i(t)$  is completely determined by  $\theta_i$ .*

Given a preference  $\succ_i$ , we define agent  $i$ 's preferences over the agenda  $\tilde{X} \in A(X)$ , noted  $\tilde{\succ}_i$ , as follows:  $\forall x, y \in \tilde{X}$ ,  $x \tilde{\succ}_i y$  if and only if  $x \succ_i y$ . The maximal set associated with the pair  $(\tilde{X}, \tilde{\succ}_i)$  is  $M(\tilde{X}, \tilde{\succ}_i) = \{x \in \tilde{X} : \forall y \in \tilde{X} \setminus \{x\}, x \tilde{\succ}_i y\}$ . Notice that since preferences are strict, maximal sets are singletons. That is,  $M(\tilde{X}, \tilde{\succ}_i) = \{\tau(\tilde{\succ}_i)\}$ , where  $\tau(\tilde{\succ}_i)$  is the top-ranked alternative in  $\tilde{X}$  for  $i$  with respect to her preferences  $\tilde{\succ}_i$ .

The profile of individual preferences depends on the realized *state*  $\theta \in \Theta^n$ . That is, a preference profile on  $\tilde{X} \in A(X)$ , associated to a profile of types

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<sup>3</sup>Indifference between alternatives is not allowed. This is a quite common assumption when the set of alternatives is finite. In this paper, it is also adopted to simplify the proofs of our main results.

<sup>4</sup>As usual,  $v_l$  and  $v_{ll}$  denote the first and the second derivate of the function  $v(l_i)$ .

$\theta = (\theta_1, \dots, \theta_n) \in \Theta^n$ , is an  $n$ -tuple  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) = (\tilde{\Phi}(\theta_1), \dots, \tilde{\Phi}(\theta_n))$  in  $P(\tilde{X})^n$ , where  $\tilde{\Phi}(\cdot)$  represents the restriction of  $\Phi(\cdot)$  over  $\tilde{X}$ . We assume each agent observes  $\theta$ , so there is complete information among the agents about their preferences over  $\tilde{X}$ . Extending our earlier conventions to preference profiles,  $\tilde{\succ}_{-i} = (\tilde{\succ}_1, \dots, \tilde{\succ}_{i-1}, \tilde{\succ}_{i+1}, \dots, \tilde{\succ}_n)$ ,  $(\hat{\succ}_i, \tilde{\succ}_{-i}) = (\tilde{\succ}_1, \dots, \tilde{\succ}_{i-1}, \hat{\succ}_i, \tilde{\succ}_{i+1}, \dots, \tilde{\succ}_n)$  and, for any group of agents  $S \subseteq I$ ,  $(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}}) = ((\tilde{\succ}_i)_{i \in S}, (\tilde{\succ}_j)_{j \in \bar{S}})$ . Finally, given a profile  $\tilde{\succ} = (\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in P(\tilde{X})^n$ , we denote  $\Theta_I(\tilde{\succ}) = \{\theta \in \Theta : \exists i \in I \text{ such that } \tilde{\succ}_i = \tilde{\Phi}(\theta)\}$  the set of *actual* types.

These preferences can be aggregated. The input for this aggregation process is the set of *declarations* of the individuals. These declarations are intended to provide information about their true types, although their sincerity may not be ensured.

The aggregation process is represented by a social choice function. For any  $\tilde{X} \in A(X)$ , a *social choice function*  $f$  on  $P(\tilde{X})^n$  is a single-valued mapping  $f : P(\tilde{X})^n \rightarrow \tilde{X}$  that associates to each preference profile  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in P(\tilde{X})^n$  a unique outcome  $f(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in \tilde{X}$ .

We are primarily interested in aggregation procedures conducted by pairwise majority voting. Then, we focus the analysis on a particular social choice function: the *median choice rule*. Let  $m : \mathfrak{R}_+^n \rightarrow \mathfrak{R}_+$  be the *median function* on  $\mathfrak{R}_+^n$  and define it as follows. For all  $x \in \mathfrak{R}_+^n$ , denote  $m(x)$  the *median* of  $x = (x_1, \dots, x_n)$  if and only if  $|\{x_i \in \mathfrak{R}_+ : x_i \leq m(x)\}| \geq \frac{(n+1)}{2}$  and  $|\{x_j \in \mathfrak{R}_+ : m(x) \leq x_j\}| \geq \frac{(n+1)}{2}$ . Since  $n$  is odd, this function is always well-defined.

**Definition 1 (Median choice rule)** *A social choice function  $f^m$  on  $P(\tilde{X})^n$  is called the median choice rule if for all  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in P(\tilde{X})^n$ ,  $f^m(\tilde{\succ}_1, \dots, \tilde{\succ}_n) = m(\tau(\tilde{\succ}_1), \dots, \tau(\tilde{\succ}_n))$ .*

In the following sections, we study the incentive compatibility properties of the median choice rule on the domain of single-crossing preference profiles. So let us introduce now the formal definition of this condition.

**Definition 2 (Single-crossing)** *A preference profile  $(\succ_1, \dots, \succ_n)$  derived from  $\Phi : \Theta \rightarrow P(X)$  is single-crossing on  $X$  if for all  $\{x, y\} \subset X$ ,  $i, j \in I$ ,*

$$[y > x, \theta_j > \theta_i, \text{ and } y \Phi(\theta_i) x] \Rightarrow [y \Phi(\theta_j) x]. \quad SC$$

We denote  $SC(X)$  the set of all single-crossing preference profiles on  $X$  (with respect to the linear order  $\geq$ ).<sup>5</sup> Notice that the property of being single-crossing is preserved in the induced preferences. That is, if  $(\succ_1, \dots, \succ_n) \in SC(X)$  then  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in SC(\tilde{X})$ , for all  $\tilde{X} \in A(X)$ , where  $SC(\tilde{X})$  is the set of all single-crossing preference profiles on  $\tilde{X}$ .

In the political arena, single-crossing makes sense if, for example, individual types are interpreted as being different ideological characters, arranged in a left-right scale, and the alternatives as public policies to be chosen by the society. Put in this way, it says that, given any two policies, one of them more to the right than the other, the more rightist a type the more will he tend to prefer the right-wing policy over the left-wing one.

The recent interest on this restricted domain of preferences is due to the fact that, like single-peakedness, single-crossing has been shown to be sufficient to guarantee the existence of majority voting equilibria. However, apart from this fact, it should be clear that both domain conditions are independent, in the sense that neither property is logically implied by the other.

To see this more formally, let us define the single-peaked condition in the following two alternative ways. For a set  $A$ , let  $Q(A)$  denote the set of all linear orders on  $A$ .<sup>6</sup> For example, if  $A \subseteq \mathfrak{R}$ , then the usual order on the real line  $\geq \in Q(A)$ . For each ordering  $q \in Q(A)$ , define the antisymmetric part of  $q$ , noted  $q_{\sim}$ , as follows:  $a q_{\sim} b$  if and only if  $a q b$  and  $\sim [b q a]$ . For instance, the antisymmetric part of the “greater than or equal to” relation over the real numbers is the “strictly greater than” relationship  $>$ .

**Definition 3 (Single-peakedness 1)** *A profile  $(\succ_1, \dots, \succ_n) \in P(X)^n$  is single-peaked on  $X$  if there exists a linear order  $q \in Q(X)$  such that  $\forall i \in I$ ,  $\{x, y, z\} \subseteq X$ ,*

$$[x \succ_i y \text{ and } x q_{\sim} y q_{\sim} z] \Rightarrow [y \succ_i z].$$

We denote  $SP(X, q)$  the set of all single-peaked (with respect to  $q$ ) preference profiles on  $X$ . Since  $q$  is common to all these profiles, we can omit it and simply write  $SP(X)$ . Clearly, given that individual preferences are strict,  $\succ \in SP(X)$  implies  $\tilde{\succ} \in SP(\tilde{X})$ , for all  $\tilde{X} \in A(X)$ .

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<sup>5</sup>Other expressions used in the literature to denominate related preference restrictions are *hierarchical adherence*, *intermediate preferences*, *order-restriction* and *unidimensional alignment*. For more on them, see Roberts (1977), Grandmont (1978), Rothstein (1990, 1991), Gans and Smart (1996), Myerson (1996), Austen-Smith and Banks (1999) and List (2001), and the references quoted there.

<sup>6</sup>A linear order over  $A$  is a reflexive, transitive and complete binary relation between the elements of  $A$ .

Alternatively, as long as we may be interested in *neutral* social choice rules (like  $f^m$ ), we can allow  $q$  to change from one profile to another. This produces a second definition of single-peakedness:

**Definition 4 (Single-peakedness 2)** *A profile  $\succ = (\succ_1, \dots, \succ_n) \in P(X)^n$  is single-peaked on  $X$  if there exists a linear order  $q^\succ \in Q(X)$ , associated to  $\succ$ , such that  $\forall i \in I, \{x, y, z\} \subseteq X$ ,*

$$[x \succ_i y \text{ and } x q^\succ y q^\succ z] \Rightarrow [y \succ_i z].$$

We denote  $SP(X, \{q^\succ\})$  the set of all preference profiles  $\succ$  on  $X$  for which there exists an ordering  $q^\succ \in Q(X)$  such that Definition 4 is satisfied. Example 2 below shows that  $SP(X) \subset SP(X, \{q^\succ\})$ .

**Example 2** *Consider the preference profiles of Table 1 and 2, over the set of alternatives  $X = \{x, y, z\}$ .*

Table 1:

$\succ_1$	$\succ_2$	$\succ_3$
$x$	$z$	$y$
$z$	$y$	$z$
$y$	$x$	$x$

Table 2:

$\hat{\succ}_1$	$\hat{\succ}_2$	$\hat{\succ}_3$
$x$	$z$	$y$
$y$	$y$	$z$
$z$	$x$	$x$

*It is easy to verify that there is no  $q \in Q(X)$  such that  $(\succ_1, \succ_2, \succ_3)$  and  $(\hat{\succ}_1, \hat{\succ}_2, \hat{\succ}_3)$  simultaneously belong to  $SP(X, q)$ . However, both profiles are in  $SP(X, \{q^\succ\})$ , since  $(\succ_1, \succ_2, \succ_3)$  is single-peaked with respect to the linear order  $y < z < x$  and  $(\hat{\succ}_1, \hat{\succ}_2, \hat{\succ}_3)$  with respect to  $z < y < x$ .*

In words, a single-peaked profile (in both versions) is one in which the set of alternatives can be linearly ordered in such a way that each agent has a unique most preferred alternative (or *ideal point*) over this common ordering, and the individual's ranking of other alternatives falls as one moves away from his ideal point. Such profiles capture the fact that, for example, an individual may have a most preferred ideological position on some left-right political scale, and the more distant is a candidate's ideological position from his most preferred alternative the more the individual may dislike the candidate.

Going back again to the relation between single-crossing and single-peakedness, it is clear for instance in Example 1 that the profile of induced policy preferences  $(w_1, \dots, w_n)$  satisfies single-crossing on the interval  $(0, 1)$ . However, for  $h(t)$  sufficiently convex, it could violate single-peakedness. Examples 3 and 4 below also illustrate this point.

**Example 3** Suppose three agents, with types  $\theta_1 < \theta_2 < \theta_3$ , who have to choose an alternative from  $\{x, y, z\} \subset \mathfrak{R}_+$ , where  $x < y < z$ . Assume preferences are as in Table 3 below. It is easy to see that this profile is single-crossing on  $\{x, y, z\}$ . However, for any ordering of the alternatives, the profile violates single-peakedness.

**Example 4** Suppose three individuals, 1, 2 and 3, who must choose an alternative from  $\{a, b, c, d\} \subset \mathfrak{R}_+$ . Assume their preferences are given by Table 4. Then, the profile  $(\succ_1, \succ_2, \succ_3)$  is single-peaked with respect to the linear order  $c < a < b < d$ . However, if each preference ordering is associated with a different type and each agent is identified with its corresponding type, then for every ordering of the types the profile of Table 4 violates single-crossing.<sup>7</sup>

Table 3: Single-crossing.

$\Phi(\theta_1)$	$\Phi(\theta_2)$	$\Phi(\theta_3)$
$x$	$x$	$z$
$y$	$z$	$y$
$z$	$y$	$x$

Table 4: Single-peakedness.

$\succ_1$	$\succ_2$	$\succ_3$
$a$	$d$	$b$
$b$	$b$	$a$
$d$	$a$	$c$
$c$	$c$	$d$

From the perspective of the analysis of strategy-proofness, there is also a substantial difference between these two preference domains. Considering Definition 3, which is the standard definition of single-peakedness used in the literature of strategy-proofness,<sup>8</sup> it is clear that  $SP(\tilde{X})$  is a Cartesian subset of  $P(\tilde{X})^n$ , for any  $\tilde{X} \in A(X)$ .<sup>9</sup> Instead, the set of all single-crossing profiles  $SC(\tilde{X})$  over  $\tilde{X}$  is not a Cartesian product preference domain. The reason is that each individual ordering (or type) in a profile  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in SC(\tilde{X})$  may not be independent of the other orderings, as Definition 2 indicates.

This has two important implications. First of all, strategy-proofness (yet to be defined) becomes then a *conditional* property of a social choice function (see Definitions 5 and 6 below). Second, the Revelation Principle does not apply on this domain. That is, even if a social choice function were found to be (conditional) strategy-proof on  $SC(\tilde{X})$ , the mechanism implementing it cannot be a *direct* one.<sup>10</sup> We will return to this point in section 5.1.

<sup>7</sup>Notice that it violates single-crossing not only for the ordering of alternatives  $c < a < b < d$ , but also for every ordering of them.

<sup>8</sup>See, for example, Moulin (1980, 1988), Barberà and Jackson (1994), Sprumont (1995), Ching (1997), Berga and Serizawa (2000) and Barberà (2001), among others.

<sup>9</sup>Of course, Single-peakedness 2 is not a Cartesian product domain.

<sup>10</sup>Remember that a direct mechanism is a game form in which the strategy space of each agent is the set of all possible individual characteristics.

Now we define the two incentive compatibility properties we seek in a social choice function on  $SC(\tilde{X})$ . These properties are called *conditional strategy-proofness* and *conditional group strategy-proofness*. That is, we are interested in voting rules over single-crossing preferences where agents, acting individually or in groups, never have incentives to misrepresent their preferences, at least if they have the opportunity to tell the true. To capture this idea, we define these two concepts:

**Definition 5 (Conditional strategy-proofness)** *A social choice function  $f$  on  $SC(\tilde{X})$  is conditional strategy-proof if for all  $(\tilde{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$ ,  $i \in I$ , and any deviation  $\hat{\succ}_i \in P(\tilde{X})$ , either  $f(\tilde{\succ}_i, \tilde{\succ}_{-i}) \tilde{\succ}_i f(\hat{\succ}_i, \tilde{\succ}_{-i})$  or  $f(\tilde{\succ}_i, \tilde{\succ}_{-i}) = f(\hat{\succ}_i, \tilde{\succ}_{-i})$ .*

If a social choice function  $f$  is not conditional strategy-proof on  $SC(\tilde{X})$ , then there exist  $i \in I$  and  $\hat{\succ}_i \in P(\tilde{X})$ , such that for some  $\tilde{\succ}_{-i} \in P(\tilde{X})^{n-1}$  and  $i$ 's true preferences  $\tilde{\succ}_i$ ,  $f(\hat{\succ}_i, \tilde{\succ}_{-i}) \tilde{\succ}_i f(\tilde{\succ}_i, \tilde{\succ}_{-i})$ , where  $(\tilde{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$ . Then,  $f$  is said *manipulable* at  $(\tilde{\succ}_i, \tilde{\succ}_{-i})$ , by  $i$ , via  $\hat{\succ}_i$ .

As usual, Definition 5 says that a social choice function  $f$  is strategy-proof on  $SC(\tilde{X})$  if for *any* preference declaration  $\tilde{\succ}_{-i}$  the rest of the agents could make, each individual  $i \in I$  considers the outcome generated by declaring his true preferences  $\tilde{\succ}_i$ ,  $f(\tilde{\succ}_i, \tilde{\succ}_{-i})$ , at least as good as  $f(\hat{\succ}_i, \tilde{\succ}_{-i})$ , where  $f(\hat{\succ}_i, \tilde{\succ}_{-i})$  is generated by  $i$ 's deviation to any other ordering  $\hat{\succ}_i \in P(\tilde{X})$ . However, since  $SC(\tilde{X})$  is a non-Cartesian domain, the declarations of the other agents  $\tilde{\succ}_{-i}$  that can be considered in the analysis are all those that are compatible with the fact that  $i$  reports his true binary relation  $\tilde{\succ}_i$  and still remains in the admissible domain of preferences.

That is, the conditional extension of strategy-proofness we introduce for non-Cartesian preference domains requires that agents have incentives to report their true preferences only in those cases (profiles) where they have the opportunity to do so. But it puts no restrictions on the individual's declaration in the other profiles.

**Definition 6 (Conditional group strategy-proofness)** *A social choice function  $f$  on  $SC(\tilde{X})$  is conditional group strategy-proof if for all  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in SC(\tilde{X})$ , and for every coalition  $S \subseteq I$ , there does not exist a joint deviation  $\hat{\succ}_S \in P(\tilde{X})^S$  such that  $f(\hat{\succ}_S, \tilde{\succ}_{\bar{S}}) \tilde{\succ}_i f(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}})$  for all  $i \in S$ .*

In the following sections, we study how well the median choice rule performs, according to these manipulation criteria, on the domain of single-crossing preferences. But, since the main motivation to do this is to study

the strategic foundation of the Representative Voter Theorem (the “single-crossing version” of the Median Voter Theorem), let us discuss first the connection between single-crossing and order-restriction, which is the original domain where this Theorem was formulated.

### 3 Single-crossing and order-restriction

Order-restriction, first formally introduced by Rothstein (1990, 1991), is a preference restriction that has been shown to be closely related to single-crossing (Gans and Smart, 1996). Next we provide its definition and an *equivalence* theorem (up to renaming of types or alternatives) that parallels that result, but emphasizing at the same time on the differences between these two domains.<sup>11</sup>

For any two sets  $A$  and  $B$  of real numbers, let  $A \gg B$ , read “ $A$  is higher than  $B$ ”, if for every  $a \in A$  and  $b \in B$ ,  $a > b$ , where  $>$  is the antisymmetric part of the “greater than or equal to” relation over the real line.

**Definition 7 (Order-restriction)** *A profile  $(\Phi(\theta_1), \dots, \Phi(\theta_n)) \in P(X)^n$  is order-restricted on  $X$  if there exists a permutation  $\gamma_\theta : \Theta \rightarrow \Theta$ , associated to  $\theta = (\theta_1, \dots, \theta_n)$ , such that  $\forall \{x, y\} \subset X$ ,  $x \neq y$ , either*

$$\{\gamma_\theta(\theta_i) \in \Theta_I(\theta) : x \Phi(\theta_i) y\} \gg \{\gamma_\theta(\theta_j) \in \Theta_I(\theta) : y \Phi(\theta_j) x\}, \quad OR - 1$$

or

$$\{\gamma_\theta(\theta_j) \in \Theta_I(\theta) : y \Phi(\theta_j) x\} \gg \{\gamma_\theta(\theta_i) \in \Theta_I(\theta) : x \Phi(\theta_i) y\}. \quad OR - 2$$

We denote  $OR(X, \{\gamma_\theta\})$  the set of all profiles  $(\Phi(\theta_1), \dots, \Phi(\theta_n))$  on  $X$  for which there exists a permutation  $\gamma_\theta : \Theta \rightarrow \Theta$  such that Definition 7 is satisfied.

In words, a preference profile is order-restricted on  $X$  if we can order the types of the agents in such a way that for any pair of alternatives the set of types preferring one of the alternatives all lie to one side of those who prefer the other. It is important to emphasize that the ordering of types is not conditional on the pair of alternatives under consideration, while the “cut-off” types may depend on the pair. Example 5 below illustrates the concept.

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<sup>11</sup>In this section, we make definitions and proofs over  $X$ , but everything is equally valid for any  $\tilde{X} \in A(X)$ .

**Example 5 (Austen-Smith and Banks, 1999)** Consider the preferences over  $X = \{x, y, z\}$ , with the order  $x < y < z$ , for the types  $\theta_1 < \theta_2 < \theta_3$ , displayed in Table 5 bellow. This profile is order-restricted over  $X$ , since there exists a permutation  $\gamma$ , defined by  $\gamma(\theta_1) = \theta_2$ ,  $\gamma(\theta_2) = \theta_1$  and  $\gamma(\theta_3) = \theta_3$ , such that under this renaming of types:

- $\{\gamma(\theta_i) : y \Phi(\theta_i) x\} = \{\theta_3\} \gg \{\theta_1, \theta_2\} = \{\gamma(\theta_j) : x \Phi(\theta_j) y\}$ ;
- $\{\gamma(\theta_i) : z \Phi(\theta_i) x\} = \{\theta_3\} \gg \{\theta_1, \theta_2\} = \{\gamma(\theta_j) : x \Phi(\theta_j) z\}$ ;
- $\{\gamma(\theta_i) : z \Phi(\theta_i) y\} = \{\theta_2, \theta_3\} \gg \{\theta_1\} = \{\gamma(\theta_j) : y \Phi(\theta_j) z\}$ .

Table 5: Order-restriction.

$\Phi(\theta_1)$	$\Phi(\theta_2)$	$\Phi(\theta_3)$
$x$	$x$	$z$
$z$	$y$	$y$
$y$	$z$	$x$

The following results exhibit the closed relationship between order-restriction and single-crossing:

**Lemma 1** *If a preference profile  $(\succ_1, \dots, \succ_n)$ , derived from  $\Phi : \Theta \rightarrow P(X)$ , is single-crossing on  $X$  then, it satisfies order-restriction on  $X$ . That is,  $SC(X) \subset OR(X, \{\gamma_\theta\})$ .*

PROOF: Consider a profile  $\succ = (\succ_1, \dots, \succ_n) \in SC(X)$ , derived from  $\Phi : \Theta \rightarrow P(X)$ . Take any pair of distinct alternatives  $\{x, y\} \subset X$  and, without loss of generality, assume  $y > x$ . Let  $\bar{\theta}$  be the smallest type in the set  $\Theta_I(\succ)$ , with respect to the linear order  $>$ , such that  $y \Phi(\bar{\theta}) x$ . If such type does not exist, then  $x \Phi(\theta_i) y$  for all  $\theta_i \in \Theta_I$  and order-restriction follows immediately. Otherwise, define the sets  $\underline{\Theta}_I^{x,y} = \{\theta_j \in \Theta_I(\succ) : \bar{\theta} > \theta_j\}$  and  $\overline{\Theta}_I^{x,y} = \{\theta_i \in \Theta_I(\succ) : \theta_i \geq \bar{\theta}\}$ . Clearly, by definition,  $\overline{\Theta}_I^{x,y} \gg \underline{\Theta}_I^{x,y}$ . On the other hand, by  $SC$ ,  $\overline{\Theta}_I^{x,y} = \{\theta_i \in \Theta_I(\succ) : y \Phi(\theta_i) x\}$  and  $\underline{\Theta}_I^{x,y} = \{\theta_j \in \Theta_I(\succ) : x \Phi(\theta_j) y\}$ , being the last equality a consequence of the completeness of the orders  $\Phi(\cdot)$ . Therefore, taken the invariant permutation  $\gamma_\theta$ , (such that  $\gamma_\theta(\theta_i) = \theta_i$  for each  $\theta_i \in \Theta$ ), it follows that  $\{\gamma_\theta(\theta_i) \in \Theta_I(\succ) : y \Phi(\theta_i) x\} \gg \{\gamma_\theta(\theta_j) \in \Theta_I(\succ) : x \Phi(\theta_j) y\}$ . But, since the pair  $x$  and  $y$  were arbitrary chosen, by  $OR - 2$ , we have that

$(\succ_1, \dots, \succ_n) \in OR(X, \{\gamma_\theta\})$ .  $\square$

However, the converse of Lemma 1 is not true. That is,  $OR(X, \{\gamma_\theta\}) \not\subset SC(X)$ . To see this, just consider the original ordering in Example 5. As we showed, it is in  $OR(X, \{\gamma_\theta\})$ ; but it is not in  $SC(X)$  as, for example,  $z \Phi(\theta_1) y$  while  $y \Phi(\theta_2) z$ , being  $z > y$  and  $\theta_2 > \theta_1$ . Nevertheless we have the following result:

**Lemma 2** *For any profile  $\succ = (\succ_1, \dots, \succ_n) \in OR(X, \{\gamma_\theta\})$ , derived from  $\Phi : \Theta \rightarrow P(X)$ , there exists a permutation  $\bar{\gamma} : \Theta \rightarrow \Theta$  such that  $\succ^{\bar{\gamma}} \in SC(X)$ , where  $\succ^{\bar{\gamma}}$  is derived from  $\Phi : \bar{\gamma}(\Theta) \rightarrow P(X)$ .*

PROOF: Consider a profile  $\succ = (\succ_1, \dots, \succ_n) \in OR(X, \{\gamma_\theta\})$ , associated to  $\theta = (\theta_1, \dots, \theta_n)$ . Since  $\succ \in OR(X, \{\gamma_\theta\})$ , there exists a permutation  $\gamma_\theta : \Theta \rightarrow \Theta$  such that for any pair of distinct alternatives  $x, y \in X$ , say  $x < y$ , we have either  $OR - 1$  or  $OR - 2$ . In the latter case, consider  $\underline{\theta} \in \Theta_I(\succ)$ , such that  $\underline{\theta} = \min_{(\theta)} \{\theta_i \in \gamma_\theta(\Theta_I(\succ)) : y \Phi(\theta_i) x\}$ . Then,  $y \Phi(\theta_j) x$  for any  $\theta_j \in \gamma_\theta(\Theta_I(\succ))$  such that  $\theta_j \geq \underline{\theta}$ . Thus, for  $\bar{\gamma} = \gamma_\theta$ , the profile  $\succ^{\bar{\gamma}}$ , derived from  $\Phi : \bar{\gamma}(\Theta) \rightarrow P(X)$ , is in  $SC(X)$ . Instead, if  $\gamma_\theta$  is such that for  $x < y$  it verifies  $OR - 1$ , consider a permutation  $\gamma' : \gamma_\theta(\Theta) \rightarrow \gamma_\theta(\Theta)$ , such that (if  $|\gamma_\theta(\Theta)| = |\Theta| = m$ ),  $\gamma'(\theta_i) = \theta_{m-i+1}$ , for every  $\theta_i \in \gamma_\theta(\Theta)$ . This permutation just induces a reversion of the ordering in  $\gamma_\theta(\Theta)$ . Then, composing  $\gamma'$  and  $\gamma_\theta$  we have a permutation  $\bar{\gamma}$  such that on  $\bar{\gamma}(\Theta_I(\succ))$  we have  $OR - 2$  and again,  $\succ^{\bar{\gamma}} \in SC(X)$ .  $\square$

Notice that Lemmas 1 and 2 amount to an equivalence (under renaming of types in  $\Theta$ ) of single-crossing and order-restriction. In fact, a similar result follows if we enlarge the domain of single-crossing preferences in a way analogous to Single-peakedness 2, by allowing the order over either  $X$  or  $\Theta$  to change from one profile to another.

**Definition 8 (Broad single-crossing)** *A profile  $(\Phi(\theta_1), \dots, \Phi(\theta_n))$  derived from  $\Phi : \Theta \rightarrow P(X)$  is broad single-crossing over  $X$  if there exists  $q^\theta \in Q(X)$  and  $p^\theta \in Q(\Theta)$  such that  $\forall \{x, y\} \subset X, i, j \in I$ ,*

$$[y q^\theta x, \theta_j p^\theta \theta_i, \text{ and } y \Phi(\theta_i) x] \Rightarrow [y \Phi(\theta_j) x], \quad BSC$$

We denote  $BSC(X, \{q^\theta, p^\theta\})$  the set of all preference profiles  $(\Phi(\theta_1), \dots, \Phi(\theta_n))$  on  $X$  for which there exist either  $q^\theta \in Q(X)$  or  $p^\theta \in Q(\Theta)$  such that Definition 8 is satisfied. Example 6 below shows that  $SC(\tilde{X}) \subset BSC(\tilde{X}, \{q^\theta, p^\theta\})$ , which means that broad single-crossing is a larger preference domain.

**Example 6** Consider three individuals and three types,  $\theta^1$ ,  $\theta^2$  and  $\theta^3$  in  $\Theta$ . Suppose the preferences of each type over  $X = \{x, y, z\} \subset \mathfrak{R}_+$  are as in Table 6. Assume first the realized state  $\theta \in \Theta^3$  is such that  $\theta_i = \theta^{3-i+1}$ , for all  $i = 1, 2, 3$ . Then  $(\Phi(\theta_1), \Phi(\theta_2), \Phi(\theta_3)) \in BSC(X, \{q^\theta, p^\theta\})$  for either  $x < y < z$  and  $\theta_1 < \theta_2 < \theta_3$  or  $z < y < x$  and  $\theta_3 < \theta_2 < \theta_1$ . On the other hand, if the realized state is  $\hat{\theta}$ ,  $\hat{\theta}_i = \theta^i$  for all  $i = 1, 2, 3$ , then  $(\Phi(\hat{\theta}_1), \Phi(\hat{\theta}_2), \Phi(\hat{\theta}_3)) \in BSC(X, \{q^{\hat{\theta}}, p^{\hat{\theta}}\})$  for either  $z < y < x$  and  $\hat{\theta}_1 < \hat{\theta}_2 < \hat{\theta}_3$  or  $x < y < z$  and  $\hat{\theta}_3 < \hat{\theta}_2 < \hat{\theta}_1$ . However, it is clear that  $(\Phi(\theta_1), \Phi(\theta_2), \Phi(\theta_3))$  and  $(\Phi(\hat{\theta}_1), \Phi(\hat{\theta}_2), \Phi(\hat{\theta}_3))$  cannot simultaneously belong to  $SC(X)$ .

Table 6:

$\Phi(\theta^1)$	$\Phi(\theta^2)$	$\Phi(\theta^3)$
$z$	$x$	$x$
$y$	$z$	$y$
$x$	$y$	$z$

**Lemma 3** For any profile  $\succ = (\succ_1, \dots, \succ_n) \in OR(X, \{\gamma_\theta\})$ , derived from  $\Phi : \Theta \rightarrow P(X)$ , there exists a profile  $\bar{\succ} \in BSC(X, \{q^\theta, p^\theta\})$ , such that  $\succ$  and  $\bar{\succ}$  are equivalent up to two permutations  $g : \Theta \rightarrow \Theta$  and  $h : X \rightarrow X$ . In the same way, for each profile  $\bar{\succ} \in BSC(X, \{q^\theta, p^\theta\})$  there exists a profile  $\succ = (\succ_1, \dots, \succ_n) \in OR(X, \{\gamma_\theta\})$  such that  $\succ \equiv \bar{\succ}$  up to two permutations  $g' : \Theta \rightarrow \Theta$  and  $h' : X \rightarrow X$ .

PROOF:

- (i) Suppose  $\succ \in BSC(X, \{q^\theta, p^\theta\})$ . Then,  $(X, q^\theta)$  is isomorphic to  $X$  under the natural order  $\geq$ . That is, there exists  $h : X \rightarrow X$  such that, if  $x q^\theta y$ , then  $h(x) \geq h(y)$ , for any  $x, y \in X$ . On the other hand,  $(\Theta, p^\theta)$  is isomorphic to  $\Theta$  under  $\geq$ ; i.e. there exists  $g : \Theta \rightarrow \Theta$  such that, if  $\theta' p^\theta \theta''$  then  $g(\theta') \geq g(\theta'')$ , for any pair  $\theta', \theta''$  in  $\Theta$ . Therefore, since  $\succ$  verifies  $BSC$  we have that

$$[h(y) > h(x), g(\theta_j) > g(\theta_i), \text{ and } h(y) \Phi(g(\theta_i)) h(x)] \Rightarrow [h(y) \Phi(g(\theta_j)) h(x)].$$

That is,  $\bar{\succ} = (\Phi(g(\theta_1)), \dots, \Phi(g(\theta_n)))$  is single-crossing over  $h(X) = X$  under  $\geq$ , for the family of types  $g(\Theta) = \Theta$  with the corresponding natural order. Then,  $\bar{\succ} \in SC(X)$  and, by Lemma 1,  $\bar{\succ} \in OR(X, \{\gamma_\theta\})$ .

- (ii) Suppose  $\succ \in OR(X, \{\gamma_\theta\})$ . From Lemma 2 we know that there exists a permutation  $g' : \Theta \rightarrow \Theta$  such that  $\succ = (\Phi(g'(\theta_1)), \dots, \Phi(g'(\theta_n))) \in SC(X)$ . But then, if we take  $q^\theta \equiv \geq$  and  $p^\theta \equiv \geq_{g(\Theta)}$ , where  $\geq_{g(\Theta)}$  is the natural order over  $g(\Theta)$ , we have that  $\succ \in BSC(X, \{q^\theta, p^\theta\})$ .  $\square$

## 4 The Representative Voter Theorem

The domain of order-restricted (single-crossing) preferences has two properties that are extremely important for collective decision-making analysis. First, as it was already mentioned in other parts of the paper, it guarantees the existence of majority voting equilibria. Second, it offers a simple characterization of the core of the majority rule. In effect, when preferences are order-restricted, the *median type* agent in the order on  $(\theta_1, \dots, \theta_n)$ , which is unique in our framework, is *decisive* in all pairwise majority contests between alternatives in  $\tilde{X}$ , for all  $\tilde{X} \in A(X)$ .<sup>12</sup> This result is sometimes referred to as the Representative Voter Theorem (RVT) or, alternatively, as the “second version” of the Median Voter Theorem.

In this section we present formally the RVT, leaving for the next section its game-theoretic counterpart.<sup>13</sup> In order to do that, let  $f^m : OR(\tilde{X}, \{\gamma_\theta\}) \rightarrow \tilde{X}$  be the median choice rule on the domain of order-restricted preferences. The non-strategic version of the Representative Voter Theorem is as follows:

**Theorem 1** *For every  $\tilde{X} \in A(X)$  and each  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in OR(\tilde{X}, \{\gamma_\theta\})$ , associated to  $\theta = (\theta_1, \dots, \theta_n)$ ,  $f^m(\tilde{\succ}_1, \dots, \tilde{\succ}_n) = M(\tilde{X}, \Phi(\theta_r))$ , where  $\theta_r = m(\theta_1, \theta_2, \dots, \theta_n)$ .*

**PROOF:** Consider a profile  $\succ \in OR(X, \{\gamma_\theta\})$ . By Lemma 2, there exists a profile  $\succ^{\tilde{\gamma}} \in SC(X)$  that obtains by renaming the types. Take the agenda  $\tilde{X} \in A(X)$  and the restriction of  $\succ^{\tilde{\gamma}}$  to  $\tilde{X}$ , noted  $\tilde{\succ}^{\tilde{\gamma}}$ . Let  $T(\tilde{X}, \tilde{\succ}^{\tilde{\gamma}}) = \{\tau(\tilde{\succ}_1^{\tilde{\gamma}}), \dots, \tau(\tilde{\succ}_i^{\tilde{\gamma}}), \dots, \tau(\tilde{\succ}_n^{\tilde{\gamma}})\}$  be the set of individuals' top-ranked alternatives in  $\tilde{X}$  according to  $\tilde{\succ}^{\tilde{\gamma}}$ . Then, for all  $i, j \in I$ ,  $\theta_i^{\tilde{\gamma}} < \theta_j^{\tilde{\gamma}}$  implies  $\tau(\tilde{\succ}_i^{\tilde{\gamma}}) \leq \tau(\tilde{\succ}_j^{\tilde{\gamma}})$ , where  $\theta_i^{\tilde{\gamma}}, \theta_j^{\tilde{\gamma}} \in \tilde{\gamma}(\Theta)$ . Suppose not. That is, assume by contradiction there exist  $i, j \in I$  such that  $\theta_i^{\tilde{\gamma}} < \theta_j^{\tilde{\gamma}}$  while  $\tau(\tilde{\succ}_i^{\tilde{\gamma}}) > \tau(\tilde{\succ}_j^{\tilde{\gamma}})$ . Since  $\tau(\tilde{\succ}_i^{\tilde{\gamma}}) \tilde{\succ}_i^{\tilde{\gamma}} \tau(\tilde{\succ}_j^{\tilde{\gamma}})$  and  $\theta_i^{\tilde{\gamma}} < \theta_j^{\tilde{\gamma}}$ , by single-crossing, we have that  $\tau(\tilde{\succ}_i^{\tilde{\gamma}}) \tilde{\succ}_j^{\tilde{\gamma}} \tau(\tilde{\succ}_j^{\tilde{\gamma}})$ . Absurd. Therefore, it follows

<sup>12</sup>See, for example, Rothstein (1991), Myerson (1996), Gans and Smart (1996), Austen-Smith and Banks (1999) and Persson and Tabellini (2000).

<sup>13</sup>It would be important to notice that we present a simplified version of the RVT, since neither individual indifference nor an even number of voters is considered. For a complete analysis, see the references listed in footnote 12.

that  $f^m(\tilde{\succ}_1^{\bar{\gamma}}, \dots, \tilde{\succ}_n^{\bar{\gamma}}) = m(\tau(\tilde{\succ}_1^{\bar{\gamma}}), \dots, \tau(\tilde{\succ}_n^{\bar{\gamma}})) = \tau(\tilde{\succ}_r^{\bar{\gamma}}) = M(\tilde{X}, \tilde{\Phi}(\theta_r^{\bar{\gamma}}))$ , where  $\theta_r^{\bar{\gamma}} = m(\theta_1^{\bar{\gamma}}, \theta_2^{\bar{\gamma}}, \dots, \theta_n^{\bar{\gamma}})$ . Finally, notice that  $\theta_r^{\bar{\gamma}} = \theta_r$ , where  $\theta_r = m(\theta_1, \dots, \theta_n)$ , since according to the proof of Lemma 2  $\bar{\gamma}$  is either the identity (meaning that, for each  $i \in I$ ,  $\theta_i^{\bar{\gamma}} = \theta_i$ ) or it is a reversion of the original ordering (implying that, for each  $i \in I$ ,  $\theta_i^{\bar{\gamma}} = \theta_{m-i+1}$ ). In either case,  $m(\theta_1^{\bar{\gamma}}, \dots, \theta_n^{\bar{\gamma}}) = m(\theta_1, \dots, \theta_n)$ .  $\square$

In words, Theorem 1 says that, given any subset of policies  $\tilde{X} \in A(X)$ , the alternative chosen by a society with order-restricted preferences coincides with the most preferred alternative of the median type agent.<sup>14</sup> This result also holds under single-peakedness if individual preferences are symmetric, but not in other cases. Lemma 4 and Figure 1 below illustrate this point.

**Lemma 4** *If  $\succ_i$  is single-peaked on  $X$  and symmetric around  $\tau(\succ_i)$  for all  $i \in I$ , then  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in OR(\tilde{X}, \{\gamma_\theta\})$  for all  $\tilde{X} \in A(X)$ .*

PROOF: Consider a preference profile  $\succ = (\succ_1, \dots, \succ_n) \in P(X)^n$  that satisfies the hypothesis of Lemma 4. Define a permutation  $\gamma : \Theta \rightarrow \Theta$  in the following way. Consider the order (over  $X$ ) of the family  $\{\tau(\succ_i)\}_{i \in I}$ , such that for each  $i$ ,  $\tau(\succ_i) = \tau(\Phi(\theta_i))$ , for  $\theta_i \in \Theta_I(\succ)$ . Suppose that the  $j$ -th item in that order is  $\tau(\succ_i)$ . Then, set  $\gamma(\theta_i) = \theta_j$ . Since  $\tau(\succ_i)$  is unique for each  $i \in I$ ,  $\gamma$  is always well-defined. Take an agenda  $\tilde{X} \in A(X)$  and the restriction of  $\succ$  to  $\tilde{X}$ , noted  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n)$ . Without loss of generality, consider an arbitrary pair of distinct alternatives  $\{x, y\} \subseteq \tilde{X}$ , say  $x < y$ , and define the ‘‘cut point’’  $\alpha = (y - x)/2$  and the sets  $\underline{T}^{x,y} = \{\tau(\succ_j) \in T(X, \succ) : \alpha > \tau(\succ_j)\}$  and  $\overline{T}^{x,y} = \{\tau(\succ_i) \in T(X, \succ) : \tau(\succ_i) \geq \alpha\}$ . Suppose  $\overline{T}^{x,y}$  and  $\underline{T}^{x,y}$  are non-empty. Then, by the construction of  $\gamma$ , it is clear that  $\{\gamma(\theta_i) \in \Theta_I(\succ) : \tau(\succ_i) \in \overline{T}^{x,y}\} \gg \{\gamma(\theta_j) \in \Theta_I(\succ) : \tau(\succ_j) \in \underline{T}^{x,y}\}$ . Furthermore, by symmetry and single-peakedness,  $\{\gamma(\theta_i) \in \Theta_I(\succ) : \tau(\succ_i) \in \overline{T}^{x,y}\} = \{\gamma(\theta_i) \in \Theta_I(\succ) : y \Phi(\theta_i) x\}$  and  $\{\gamma(\theta_j) \in \Theta_I(\succ) : \tau(\succ_j) \in \underline{T}^{x,y}\} = \{\gamma(\theta_j) \in \Theta_I(\succ) : x \Phi(\theta_j) y\}$ . Therefore,  $\{\gamma(\theta_i) \in \Theta_I(\tilde{\succ}) : y \Phi(\theta_i) x\} \gg \{\gamma(\theta_j) \in \Theta_I(\tilde{\succ}) : x \Phi(\theta_j) y\}$ . That is,  $OR - 2$  holds. But, since  $\{x, y\}$  was arbitrary chosen, it follows that  $\tilde{\succ} \in OR(\tilde{X}, \{\gamma_\theta\})$ . On the other hand, if either  $\overline{T}^{x,y}$  or  $\underline{T}^{x,y}$  is empty, then order-restriction follows immediately, since either  $x \tilde{\Phi}(\theta) y$  for all  $\theta \in \Theta_I(\tilde{\succ})$

<sup>14</sup>Rothstein (1991) has also shown that, when preferences are strict and the number of voters is odd, as in our case, the preference ordering generated by the majority rule coincides with the preference relation associated to the median type agent. This implies that the majority preference relation inherits all the properties of the median type agent’s preference ordering. In particular, transitivity. Gans and Smart (1996) have proven a similar result for non-strict preference orderings, but under strict single-crossing.

or  $y \tilde{\Phi}(\theta) x$  for all  $\theta \in \Theta_I(\tilde{\succ})$ .  $\square$

Figure 1 below exhibits a case where individual preferences are single-peaked, but they are not symmetric around the top. The picture shows then that the decisive agent depends on the particular agenda analyzed. In effect, consider first the full set of alternatives  $X = [0, 1]$ . Since preferences are single-peaked and  $|I|$  is odd, the Median Voter Theorem says agent 2's *unrestricted* top,  $\tau(\succ_2)$ , is the unique majority voting equilibrium. Now consider the subset  $\tilde{X} = \{a, b, c, d\}$ . The *induced* profile is still single-peaked (along the linear ordering  $c < a < b < d$ ).<sup>15</sup> However, it is not single-crossing. Therefore, it turns out that agent 2's most preferred alternative in  $\tilde{X}$ ,  $d$ , is defeated by  $b$ , which is agent 3's *restricted* top and the Condorcet winner in  $\tilde{X}$ . This negative result does not hold under order-restriction (single-crossing). As we have seen, Theorem 1 guarantees that the individual who has the median type is decisive over any non-empty subset.

[Insert Figure 1 about here]

However, is the outcome predicted by the RVT robust to individual or group manipulation? That is, can we expect the society to select this alternative when individuals act strategically? The Representative Voter Theorem is a result derived under the assumption that individuals honestly reveal their preferences or, alternatively, under the assumption that the decision-maker knows them. Both assumptions are obviously very strong.

Fortunately, it turns out that even if we relax them, admitting both private information of individual values and strategic behavior on the part of voters, the RVT still holds. As we will see, the reason is that the median choice rule is strategy-proof on the domain of single-crossing preference profiles. This implies that, in any majority contest, each agent has a dominant strategy, which is to honestly reveal his preferences. In the next section, we derive this result formally and we provide an *indirect* mechanism that implements the prediction of the RVT in dominant strategies.

## 5 Manipulation over single-crossing domains

The manipulation of the median rule has been studied for a long time in the literature of social choice. The earliest reference goes back to the seminal

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<sup>15</sup>In fact, it is the profile introduced in Example 4 above.

paper of Black (1948). Since then, a lot of progress has been made towards the understanding of its properties. For instance, it is well-known today that there exists a preference domain where this voting procedure performs quite well, in terms of its capacity to extract truthful information about the preferences of the agents. This domain is of course single-peakedness.

In this section, we analyze whether the median choice rule can be manipulated on a different preference domain, namely over single-crossing preferences. Even though this family of preferences is now employed in many models of collective decision-making, nothing has been said in the literature about the possibility of manipulation over this domain. In particular, the “single-crossing version” of the Median Voter Theorem (the RVT) is used without caring much about its strategic foundation. The main purpose here is therefore to fill out this gap.

Our main result is the following:

**Proposition 1** *The median choice rule  $f^m$  is conditional strategy-proof over  $SC(\tilde{X})$ , for any  $\tilde{X} \in A(X)$ .*

PROOF: Consider a profile  $(\tilde{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$ , where agent  $i$ , of type  $\theta_i$ , has preferences  $\tilde{\succ}_i$ . Suppose that there exists another type  $\hat{\theta}_i$  such that  $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) \tilde{\succ}_i f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$ , where  $\hat{\succ}_i = \tilde{\Phi}(\hat{\theta}_i)$ . Furthermore, without loss of generality, assume  $\tau(\tilde{\succ}_i) < f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$ . We have two cases to consider:

1.  $\tau(\hat{\succ}_i) \leq f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$ . Then  $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) = f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$ . Contradiction;
2.  $\tau(\hat{\succ}_i) > f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$ . Then  $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) > f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$ . Denote  $\tilde{\tau} = f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$  and  $\hat{\tau} = f^m(\hat{\succ}_i, \tilde{\succ}_{-i})$ . Since we assume  $(\tilde{\succ}_i, \tilde{\succ}_{-i})$  verifies the single-crossing property, we have that  $\hat{\tau} \tilde{\Phi}(\theta) \tilde{\tau}$  for all  $\theta \geq \theta_i$ . On the other hand, since  $\tilde{\tau}$  is the top for at least one ordering, say  $\tilde{\succ}_j$ , in the profile  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n)$ , it must be that the type corresponding to  $\tilde{\succ}_j$ ,  $\theta_j$ , is such that  $\theta_j < \theta_i$ . But then, since  $\tau(\tilde{\succ}_i) < \tilde{\tau}$ , by single-crossing we have that  $\tilde{\tau} \tilde{\Phi}(\theta) \tau(\tilde{\succ}_i)$  for every  $\theta \geq \theta_j$ . In particular for  $\theta_i$ . Contradiction.  $\square$

Thus Proposition 1 makes the contribution of proving that, in addition to single-peakedness, there exists another very *natural* preference domain over the real line where strategy-proof social choice rules can be found. That is, it shows that single-crossing constitutes a domain restriction that guarantees not only majority voting equilibria, but also the existence of non-trivial

strategy-proof social choice functions. In particular, this is true for the median choice rule.<sup>16</sup>

Now we extend the result of Proposition 1 by showing that in fact these conditions, i.e., single-crossing and broad single-crossing, assure strategy-proofness not only at individual level but also at group level.

**Proposition 2** *The median choice rule  $f^m$  is conditional group strategy-proof over  $SC(\tilde{X})$ , for any  $\tilde{X} \in A(X)$ .*

PROOF: Consider a profile  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in SC(\tilde{X})$ , with associated types  $(\theta_1, \dots, \theta_n)$ . Suppose there exists a coalition  $S \subseteq I$  and a list of alternative types for members of  $S$ ,  $(\hat{\theta}_i)_{i \in S}$ , such that  $f^m(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}}) \tilde{\succ}_i f^m(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}})$  for all  $i \in S$ , where  $\hat{\succ}_S = (\tilde{\Phi}(\hat{\theta}_i))_{i \in S}$ . For simplicity, denote  $\tilde{\tau} = f^m(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}})$  and  $\hat{\tau} = f^m(\hat{\succ}_S, \tilde{\succ}_{\bar{S}})$ . Notice that, by the definition of  $f^m$ ,  $\tilde{\tau}$  and  $\hat{\tau}$  coincide with the tops corresponding to the orderings reported by some voters. Denote these agents  $j$  and  $j'$  and their types  $\theta_j$  and  $\theta_{j'}$ , respectively. Since  $\tilde{\tau} \neq \hat{\tau}$ , assume  $\tilde{\tau} < \hat{\tau}$ . Then, for all  $i \in S$ ,  $\tau(\tilde{\succ}_i) > \tilde{\tau}$ . Suppose not. That is, assume  $\tau(\tilde{\succ}_i) \leq \tilde{\tau}$  for some agent  $i \in S$ . If  $\tau(\tilde{\succ}_i) = \tilde{\tau}$ , then  $\tilde{\tau} \tilde{\succ}_i \hat{\tau}$ , which contradicts our initial hypothesis. Consider, instead, that  $\tau(\tilde{\succ}_i) < \tilde{\tau}$ . Since  $\hat{\tau} \tilde{\succ}_i \tilde{\tau}$ , by single-crossing we have that for all  $\theta \geq \theta_i$ ,  $\hat{\tau} \tilde{\Phi}(\theta) \tilde{\tau}$ . Then,  $\theta_j$  has to verify  $\theta_j < \theta_i$ ; and, by single-crossing,  $\tilde{\tau} \tilde{\Phi}(\theta_j) \tau(\tilde{\succ}_i)$  implies  $\tilde{\tau} \tilde{\Phi}(\theta_i) \tau(\tilde{\succ}_i)$ . Contradiction. Then,  $\tau(\tilde{\succ}_i) > \tilde{\tau}$  for all  $i \in S$ . The rest of the proof is as follows. By definition,

$$f^m(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}}) = m(\tau(\tilde{\succ}_1), \dots, \tau(\tilde{\succ}_n)) = \tilde{\tau},$$

while

$$f^m(\hat{\succ}_S, \tilde{\succ}_{\bar{S}}) = m(\{\tau(\hat{\succ}_i)\}_{i \in S}, \{\tau(\tilde{\succ}_j)\}_{j \in \bar{S}}) = \hat{\tau}.$$

Two cases are possible:

1. For each  $i \in S$ ,  $\tau(\hat{\succ}_i) > \tilde{\tau}$ . Then  $\hat{\tau} = \tilde{\tau}$ . Contradiction.

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<sup>16</sup>In Saporiti and Tohmé (2003), it is shown that the whole family of conditional strategy-proof social choice functions over single-crossing preferences is given by a subclass of the *extended median rules*, obtained by distributing the fixed parameters (also called *phantom voters*) at the extremes of the real line. This subclass, where each phantom is either a *leftist* or a *rightist*, is sometimes referred to as *positional dictator* choice rules (see Moulin (1988), pp. 302). They select the  $k$ th ranked peak among the tops of the reported orderings, for some  $k = 1, \dots, n$ . For example, if  $k = 1$ , we have the *leftist rule*, which chooses the smallest reported peak of a real voter. Of course, the median choice rule is also a particular case.

2. For some  $i \in S$ ,  $\tau(\hat{\succ}_i) \leq \tilde{\tau}$ . Then, by rewritten  $(\{\tau(\hat{\succ}_i)\}_{i \in S}, \{\tau(\tilde{\succ}_j)\}_{j \in \bar{S}})$  as  $(y_1, \dots, y_n)$ , we have that

$$\left| \{j \in \{1, \dots, n\} : y_j \leq \tilde{\tau}\} \right| \geq \frac{(n+1)}{2}.$$

But this implies that  $m(y_1, \dots, y_n) \leq \tilde{\tau}$ . That is,  $f^m(\hat{\succ}_S, \tilde{\succ}_{\bar{S}}) \leq f^m(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}})$ . Contradiction.  $\square$

**Corollary 1**  $f^m$  is conditional group strategy-proof over  $BSC(\tilde{X}, \{q^\theta, p^\theta\})$ , for any  $\tilde{X} \in A(X)$ .<sup>17</sup>

PROOF: Assume, by contradiction, that  $f^m$  is manipulable at some profile  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in BSC(\tilde{X}, \{q^\theta, p^\theta\})$ . By the same argument applied in part (i) of the Proof of Lemma 3, the profile  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n)$  is equivalent, up to permutations  $h : \tilde{X} \rightarrow \tilde{X}$  and  $g : \Theta \rightarrow \Theta$ , to a profile  $(\tilde{\succ}'_1, \dots, \tilde{\succ}'_n)$  which is also single-crossing over  $h(\tilde{X}) = \tilde{X}$  under  $\geq$ , for the family of types  $g(\Theta) = \Theta$  with the corresponding natural order. Then,

$$f^m(\tilde{\succ}') = m_{\geq}(\tau(\tilde{\succ}'_1), \dots, \tau(\tilde{\succ}'_n)) = m_{\geq}(h \circ \tau(\tilde{\succ}_{g(1)}), \dots, h \circ \tau(\tilde{\succ}_{g(n)})),$$

where  $m_{\geq}(\cdot)$  is the median under the order  $\geq$ . Notice that by the properties of both the median and the permutation  $g$ , we have that  $m_{\geq}(h \circ \tau(\tilde{\succ}_{g(1)}), \dots, h \circ \tau(\tilde{\succ}_{g(n)})) = m_{\geq}(h \circ \tau(\tilde{\succ}_1), \dots, h \circ \tau(\tilde{\succ}_n))$ . Then, we can apply the proof of Proposition 2, to show that  $m_{\geq}(h \circ \tau(\tilde{\succ}_1), \dots, h \circ \tau(\tilde{\succ}_n))$  is not group-manipulable over  $(\tilde{X}, \geq)$ . Therefore, no group  $S \subseteq I$  exists, such that each  $i \in S$  benefits from declaring an alternative  $\hat{\succ}_i$ , such that  $\tau(\tilde{\succ}_i) \neq \tau(\hat{\succ}_i)$ . Contradiction.  $\square$

**Corollary 2**  $f^m$  is conditional group strategy-proof over  $OR(\tilde{X}, \{\gamma_\theta\})$ , for any  $\tilde{X} \in A(X)$ .

PROOF: Immediate from Lemma 3 and Corollary 1.  $\square$

Next we use these positive results to provide the game-theoretic counterpart of the Representative Voter Theorem. But before that, remember that, according to the Revelation Principle, strategy-proofness is a *necessary* condition for truthful or *direct* implementation. However, it is not *sufficient*. It is sufficient when the preference domain of the social choice function is a

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<sup>17</sup>The definition of conditional group strategy-proof for  $f^m$  over  $BSC(\tilde{X}, \{q^\theta, p^\theta\})$  follows from Definition 6, by changing in the appropriate way the domain of the choice rule. The same applies in Corollary 2 below for  $OR(\tilde{X}, \{\gamma_\theta\})$ .

Cartesian product. When it is not, the direct mechanism is not well-defined, since the set of strategies of each agent, i.e., the set of admissible individual preferences that can be declared, depends on the strategies used by the others.<sup>18</sup>

This is precisely our case. Proposition 1 shows that  $f^m$  is strategy-proof over  $SC(\tilde{X})$ , for any  $\tilde{X} \in A(X)$ . Therefore, the necessary condition for the application of the Revelation Principle holds. However, the sufficient condition fails, since  $SC(\tilde{X})$  is a non-Cartesian domain. Hence, the implementation of  $f^m$  has to be explicitly analyzed.

## 5.1 Implementation of the median choice rule

In this section, we propose an extensive game form that can be used to *indirectly* implement  $f^m$  in dominant strategies. We show that this game form is essentially equivalent to a *reduced* mechanism in normal form, and we prove that the latter succeeds in implementing the median rule. We also briefly discuss why the extensive game form or its associated reduced game form works, but not a direct mechanism in which each individual simply declares his top in  $\tilde{X}$ . Finally, we derive the game-theoretic equivalent of Theorem 1.

Suppose individuals have preferences  $(\succ_1, \dots, \succ_n) \in SC(X)$ . Assume the selection of a social outcome in the agenda  $\tilde{X} \in A(X)$ , which is the planner's basic problem, is indirectly performed by the following two-stage voting procedure. In the first stage, individuals select by pairwise majority voting a *representative* individual from the set  $I$ . Then, in the second stage, the winner chooses an alternative in  $\tilde{X}$ , which is then the policy implemented by the planner.

Clearly, in the last stage each individual  $i$  has a dominant strategy, which is simply to choose his most preferred alternative in  $\tilde{X}$ . Therefore, it is immediate to see that the extensive game form is equivalent to a *reduced* strategic game form in which individuals choose by pairwise majority comparisons an alternative from the set of actual ideal points  $T(\tilde{X}, \tilde{\succ}) = \{\tau(\tilde{\succ}_1), \dots, \tau(\tilde{\succ}_n)\}$ , where  $\tilde{\succ} = (\tilde{\succ}_1, \dots, \tilde{\succ}_n)$  is the restriction of  $(\succ_1, \dots, \succ_n)$  to  $\tilde{X}$ . We prove that this reduced mechanism can be used to implement  $f^m$  in a dominant strategy equilibrium.

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<sup>18</sup>A possible way of solving this consists in asking to each agent to report a preference profile, instead of his individual preference ordering. If the social choice function is strategy-proof, then it can be shown that reporting the true preferences of the whole society is a dominant strategy for each individual. See Osborne and Rubinstein (1994) for a proof.

**Definition 9** A mechanism  $\Gamma$  with consequences in  $\tilde{X}$  is a strategic game form  $\langle I, (S_i), \phi \rangle$ , where  $I$  is the set of players,  $S_i$  is the set of actions available for each agent  $i \in I$ , and  $\phi : \prod_{i \in I} S_i \rightarrow \tilde{X}$  is an outcome function that associates an alternative with every action profile.

We say that  $\Gamma$  implements a social choice function  $f : P(\tilde{X})^n \rightarrow \tilde{X}$  in dominant strategies if there exists a dominant strategy equilibrium for the mechanism, yielding the same outcome as  $f$  for each possible preference profile  $\tilde{\succ} \in P(\tilde{X})^n$ . This is formally stated in Definition 10.

**Definition 10** The mechanism  $\Gamma = \langle I, (S_i), \phi \rangle$  implements the social choice function  $f : P(\tilde{X})^n \rightarrow \tilde{X}$  in dominant strategies if there exists a dominant strategy equilibrium of  $\Gamma$ , noted  $s^*(\cdot) = (s_1^*(\cdot), \dots, s_n^*(\cdot))$ , such that  $\phi(s^*(\tilde{\succ})) = f(\tilde{\succ})$  for all  $\tilde{\succ} \in P(\tilde{X})^n$ .

**Proposition 3** There exists a mechanism that implements  $f^m : SC(\tilde{X}) \rightarrow \tilde{X}$  in dominant strategies over  $\tilde{X}$ .

PROOF: Consider a preference profile  $\tilde{\succ} \in SC(\tilde{X})$  and the mechanism  $\Gamma = \langle I, (S_i), \phi \rangle$ , where  $I$  is the set of players; an action for agent  $i \in I$  is simply to choose an element in  $S_i = T(\tilde{X}, \tilde{\succ})$ ; and the outcome function  $\phi(s_1, \dots, s_n) = m(s_1, \dots, s_n)$ . We show that the action profile  $(\tau(\tilde{\succ}_1), \dots, \tau(\tilde{\succ}_n))$  constitutes a dominant strategy equilibrium of the game induced by  $\Gamma$ . That is,

$$\phi(s_1, \dots, \tau(\tilde{\succ}_i), \dots, s_n) \tilde{\succ}_i \phi(s_1, \dots, \hat{s}_i, \dots, s_n)$$

for all  $i \in I$ ,  $\hat{s}_i \neq \tau(\tilde{\succ}_i)$ ,  $s_{-i} \in \prod_{j \neq i} S_j$ . Since, by definition,  $\phi(\cdot) = m(\cdot)$ , we can easily recast the proof of Proposition 1 to fit in this scheme. Suppose that there exists such  $\hat{s}_i$ . Call  $\tilde{s} = \phi(\tau(\tilde{\succ}_i), s_{-i})$  and  $\hat{s} = \phi(\hat{s}_i, s_{-i})$ . Without loss of generality, assume  $\tau(\tilde{\succ}_i) < \tilde{s}$ . We have two cases to consider:

1.  $\hat{s}_i \leq \tilde{s}$ . Then  $m(\tau(\tilde{\succ}_i), s_{-i}) = m(\hat{s}_i, s_{-i})$ . Therefore,  $\phi(\tau(\tilde{\succ}_i), s_{-i}) = \phi(\hat{s}_i, s_{-i})$ . Contradiction.
2.  $\hat{s}_i > \tilde{s}$ . Then the new median  $\hat{s}$  will be in the interval  $(\tilde{s}, \hat{s}_i]$ . By hypothesis,  $\hat{s} \tilde{\succ}_i \tilde{s}$ . Furthermore, since the preferences are single-crossing on  $T(\tilde{X}, \tilde{\succ})$  and  $\hat{s} > \tilde{s}$ , for every  $\theta \geq \theta_i$  we have that  $\hat{s} \tilde{\Phi}(\theta) \tilde{s}$ . On the other hand, notice that, since each  $S_j = T(\tilde{X}, \tilde{\succ})$ , there must exist  $\theta_j \in \Theta_I(\tilde{\succ})$  such that  $\tilde{s} = \tau(\tilde{\Phi}(\theta_j))$ . Moreover,  $\theta_j$  must be such that  $\theta_j < \theta_i$ . But then, since  $\tau(\tilde{\succ}_i) < \tilde{s}$  and  $\tilde{s} \tilde{\Phi}(\theta_j) \tau(\tilde{\succ}_i)$ , by single-crossing, we have that  $\tilde{s} \tilde{\Phi}(\theta) \tau(\tilde{\succ}_i)$  for all  $\theta \geq \theta_j$ ; in particular for  $\theta_i$ . Contradiction.

Therefore,  $(\tau(\tilde{\succ}_1), \dots, \tau(\tilde{\succ}_n))$  is a dominant strategy equilibrium.  $\square$

The fact that the alternative declared by each agent is restricted to belong to  $T(\tilde{X}, \tilde{\succ})$ , the set of all individual maximal alternatives in  $\tilde{X}$ , is crucial for the proof of Proposition 3. It is easy to see that a mechanism based on direct declarations of the most preferred alternatives in  $\tilde{X}$  cannot be used to implement  $f^m$ . For instance, in Example 3, if agents are asked to declare their most preferred alternatives in  $\tilde{X} = \{x, y, z\}$ , a manipulation cannot be avoided. If agent 1 and agent 3 declare  $y$  and  $z$ , respectively, then player 2 will prefer to announce  $z$  instead of his true top  $x$ .<sup>19</sup>

Instead, our *indirect* mechanism works because the induced preferences over the set  $T(\tilde{X}, \tilde{\succ})$ , derived from  $\tilde{\succ} \in SC(\tilde{X})$ , are single-peaked. This is formally shown in Lemma 5 below.

**Lemma 5** *If a preference profile  $\tilde{\succ} = (\tilde{\succ}_1, \dots, \tilde{\succ}_n)$  is single-crossing over  $\tilde{X}$ , then the restriction of  $\tilde{\succ}$  over the set  $T(\tilde{X}, \tilde{\succ})$  is single-peaked.*

PROOF: For a given profile  $(\tilde{\succ}_1, \dots, \tilde{\succ}_n) \in SC(\tilde{X})$  and the associated set  $T(\tilde{X}, \tilde{\succ})$ , consider the restriction of  $\tilde{\succ}$  to  $T(\tilde{X}, \tilde{\succ})$ , denoted  $\tilde{\succ}^T = (\tilde{\succ}_1^T, \dots, \tilde{\succ}_n^T)$ . By contradiction, suppose  $\tilde{\succ}^T \notin SP(T)$ , where  $SP(T)$  is the set of all single-peaked preference profiles over  $T(\tilde{X}, \tilde{\succ})$  (with respect to the linear order  $\leq$ ). Then, there exist an individual  $i \in I$ , with type  $\theta_i \in \Theta$ , and  $x, y, \tau(\tilde{\succ}_i) \in T(\tilde{X}, \tilde{\succ})$  such that

$$x < y \leq \tau(\tilde{\succ}_i), \text{ but } x \tilde{\succ}_i^T y.$$

Thus,  $y \neq \tau(\tilde{\succ}_i)$ . Moreover, since  $\tilde{\succ}^T \in SC(T)$ ,  $x \tilde{\succ}_j^T y$  for all  $\theta_j \leq \theta_i$ . This means  $y \neq \tau(\tilde{\succ}_j)$  for all  $\theta_j \leq \theta_i$ . However, since we assume  $y \in T(\tilde{X}, \tilde{\succ})$ , then  $y = \tau(\tilde{\succ}_k)$  for some individual  $k \in I$ , with type  $\theta_k > \theta_i$ . Then,  $y \tilde{\succ}_k \tau(\tilde{\succ}_i)$  implies  $y \tilde{\succ}_j \tau(\tilde{\succ}_i)$  for all  $\theta_j \leq \theta_k$ . In particular, for  $\theta_i$ . Contradiction. The same argument applies if  $\tau(\tilde{\succ}_i) \leq y < x$  and  $x \tilde{\succ}_i^T y$ . Hence,  $\tilde{\succ}^T \in SP(T)$ .  $\square$

It is easy to show that the converse of Lemma 5 does not hold. That is, preferences can be single-peaked over  $T(\tilde{X}, \tilde{\succ})$ , but not necessarily single-crossing on  $T(\tilde{X}, \tilde{\succ})$ . The preference profile of Table 7 below illustrates this.

Finally, we derive the following corollaries from Proposition 3:

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<sup>19</sup>Proposition 1 shows that individual manipulation is ruled out when agents are required to declare a complete preference ordering, and not just the top alternative. The intuition is again illustrated by Example 3. Notice that in this case individual 1 cannot submit an ordering with the alternative  $y$  as its top without violating the single-crossing condition. Thus, player 2 has no reason to lie.

Table 7: Counterexample

$\gamma_1$	$\gamma_2$	$\gamma_3$	$\gamma_4$
$w$	$x$	$y$	$z$
$x$	$y$	$x$	$y$
$y$	$z$	$w$	$x$
$z$	$w$	$z$	$w$

**Corollary 3** For any  $\tilde{X} \in A(X)$ , there exists a mechanism that implements  $f^m : OR(\tilde{X}, \{\gamma_\theta\}) \rightarrow \tilde{X}$  in dominant strategies over  $\tilde{X}$ .

PROOF: Consider any preference profile  $\tilde{\succ} \in OR(\tilde{X}, \{\gamma_\theta\})$ . By Lemma 2, there exists a permutation  $\tilde{\gamma}$  of  $\Theta$  that generates a profile  $\tilde{\succ}^{\tilde{\gamma}} \in SC(\tilde{X})$ . Hence, the mechanism defined in Proposition 3 yields, as the outcome of its dominant strategy equilibrium, the median value of the maximal alternatives over  $\tilde{X}$ ,  $\phi(\tilde{\succ}^{\tilde{\gamma}}) = m(\tau(\tilde{\succ}_1^{\tilde{\gamma}}), \dots, \tau(\tilde{\succ}_n^{\tilde{\gamma}})) = \tau(\tilde{\Phi}(\theta_r^{\tilde{\gamma}}))$ . But, this outcome coincides with  $f^m(\tilde{\succ})$  because, as seen in Theorem 1,  $m(\theta_1^{\tilde{\gamma}}, \dots, \theta_n^{\tilde{\gamma}}) = m(\theta_1, \dots, \theta_n)$ .  $\square$

**Corollary 4** For any  $\tilde{X} \in A(X)$ , there exists a mechanism that implements  $f^m : BSC(\tilde{X}, \{q^\theta, p^\theta\}) \rightarrow \tilde{X}$  in dominant strategies over  $\tilde{X}$ .

PROOF: Immediate from Lemma 3 and Corollary 3.  $\square$

Corollary 3 provides the *strategic counterpart* of Theorem 1. That is, it shows that, when preferences are order-restricted, the social outcome under pairwise majority voting, i.e. the most preferred alternative of the median type, can be attained by a reduced mechanism in which each agent is allowed to choose an alternative among the top-ranked alternatives in the feasible set of policies. Or, equivalently, it can be achieved by following a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

## 6 Final remarks

In this paper we present several results. First of all, we prove that, in addition to single-peakedness, there exists another *natural* preference domain over the

real line for which strategy-proof social choice rules can be found. Concretely, we show that single-crossing (and its extension broad single-crossing) constitutes a domain restriction that allows not only majority voting equilibria, but also the existence of non-trivial strategy-proof (as well as group-strategy-proof) social choice functions. In particular, this is true for the median rule (Propositions 1 and 2 and Corollaries 1 and 2).

The main feature to remark of this result is that single-crossing preferences do not necessarily satisfy single-peakedness, and vice versa (Examples 3 and 4). But, as it is known, in one-dimensional collective decision models, the latter is the most frequent domain restriction used to guarantee strategy-proofness. Thus, our result shows that, at least for the median rule, the existence of a *particular* kind of interdependence between individuals' types (preferences) is also sufficient to ensure non-manipulation both at the individual and at the group level.

The second important group of results is given by Lemmas 1-3, which exhibit the close relation between single-crossing and order-restriction. A previous work in the same direction is Gans and Smart (1996), in which these preference domains are shown to be essentially equivalent. Nevertheless, our results differ from theirs in two ways. First, ours seem to be more consistent with Rothstein's original characterization of order-restriction. Second, particular attention is devoted here to the fact that these conditions may not be *directly* equivalent. The crucial point to understand this difference is that, unlike single-crossing, order-restriction does not assume any ordering on the set of possible alternatives. Furthermore, it is precisely this feature that make order-restriction interesting for analyzing strategy-proofness in multi-dimensional choice spaces and over restricted agendas.

Finally, these results are used at the end of the paper to show that the Representative Voter Theorem has a well-defined non-cooperative strategic foundation. We show that the collective outcome predicted by the RVT can be implemented through a simple sequential mechanism. This mechanism is a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner (Proposition 3 and Corollaries 3 and 4). Given that the structure of this mechanism presents some features that we observe frequently in "real" voting processes, the analysis carried out here may also provide insights for a rationale of these "real" voting situations.

At the same time, there are significant issues that this paper does not cover. The most important one that we have left for a future work is to fully characterize the family of strategy-proof social choice functions over single-crossing preferences. Of course, the classes that also satisfy other requirements like anonymity, Pareto efficiency or combinations of them should

also be determined.

On the other hand, another relevant question that we do not address here is how these results change when individual preferences are allowed to express indifference between alternatives. Our simplification is partially justified by the fact that the set of possible outcomes is finite. However, we guess substantial changes may be expected if this assumption is dropped.

Finally, another problem that should also be analyzed is how these conditions, i.e., single-crossing and order-restriction, can be extended to multidimensional spaces. That is, it should be considered the way in which these preference restrictions can deal with both multidimensional choice sets and conflicts of interests that cannot be projected onto a one-dimensional space.

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