

Problems of Voting

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It is now over 200 years since Condorcet discovered a serious paradox in the standard method of voting under substantially any of the rules of order. Most Americans are most familiar with the rules developed by General Robert's, but almost all sets of rules used in genuinely democratic bodies are similar. A proposal is made and then modifications, called amendments are added. There is then a series of binary votes so arranged that the final outcome is a modified version of the original proposal. The modification can be extreme. Unfortunately, Condorcet discovered that some amendments or proposals which had been defeated in the earlier part of the voting series, might be able to beat the ultimate outcome.

A colleague of Condorcet in the Academy proposed another voting method in which all alternatives are listed by each voter and then given a weight depending upon their location in the list. It was immediately pointed out that this method provided strategic opportunities for the voter to cheat by misspecifying his preferences. Borda replied that he designed his system for honest men, but since politics is not full of honest men this was regarded as a weak defense.

The problem was largely ignored for the next two centuries. Some mathematicians seem to have been interested and one of them Dodgson, better known as Lewis Carroll put in a good deal of work on the matter, but without reaching any solution. People advocating democracy, or for that matter opposing it, seem not to have know about this problem or in any event they did not discuss it.

In the 1930s a mathematician interested in field sports and engaged in scoring sports meets discovered that the method used for scoring in which the winner in a given sport was given five points, the second three points, and the third one point, and these points were aggregated in order to determine which university team won had a serious paradox¹. Specifically whether A or B won the meet could be determined by whether C, a weak team was physically present or not. He published this result in a mathematical journal and it was ignored for some time. It was not until the late 40s that Arrow used this as part of his proof that voting methods all have severe difficulties.

Arrow listed listed five criteria for an ideal voting method, the first two of which were

¹ E.V. Huntingdon, A Paradox in the Scoring of Competing teams, Science, Vol 88, Sept 23, 1938, pp. 287-88

simply that the election be logically coherent and the fifth was that it be a voting method and not a dictatorship. The third, however, was the absence of intransitivity which Condorcet had demonstrated could occur following the rules of order by taking the proposals in a ordered series of pair wise votes. The fourth was that choice between two alternatives should not be affected by consideration of a third alternative. Below I shall show that the Borda method fails this last criterion and indeed any voting method which in the final choice of the winning alternative takes into account not only one other alternative, but more than one, will not meet this criterion. This is called dependence upon an irrelevant alternative, and I will argue, it may not be undesirable.

Shortly before Arrow worked on this problem Black and Newing submitted a completely different proof that voting normally has severe paradoxes². This proof, which will be shown below, and is even less widely known than the Arrow proof, was submitted to *Econometrica*, where it encountered almost pathologically bad refereeing. While it was undergoing a long delay Arrow produced his proof and it was published³. When the referee report finally arrived, Black and Newing were told that they would have to discuss Arrow's proof. They refused and withdrew the article. It was eventually published as part of a book in 1960.

I should deviate from the main subject of mathematics here to point out that there's no reason to even suspect Arrow of plagiarism. Not only is he honest, but the two proofs have substantially no resemblance. They reach the same destination by radically different routes. Further in a way the Arrow proof is more general, since it deals with the Borda and many other methods as well as the standard rules of order procedure⁴.

Black and Newing introduced, however, an additional item here. They pointed out that if all of the alternatives could be arranged along a single line so that each voter favored that alternative closest to his optimum over any other farther away, then the paradox did not occur. Arrow accepted this in later writings and I think most people have accepted it. It looks obvious, but I shall demonstrate below that it is of only trivial importance.

The voting problem attracted a good deal of attention in the first few years after Arrow's publication, but has largely disappeared from the literature recently. Carl J Dahlman in an extremely good discussion of the whole problem⁵

² Black, Duncan & Newing R.A. "Committee Decisions with Complementary Valuation" London, William Hodge, 1951

³ Arrow Kenneth, "A Difficulty in the Concept of Social Welfare" *Journal of Political Economy*, Vol 58, No 4, Aug. 1950 pp 328-346.

⁴ Professor Gordon Brady made a careful study of the matter and wrote an unpublished essay on it. He can be reached at GBrady6430@aol.com.

⁵ "between the rule of law and the rule of public choice: on Arrowian impossibility institutions" unpublished, but available from the Hoover institution and on the Web. Dahlman@hoover.Stanford.Edu.

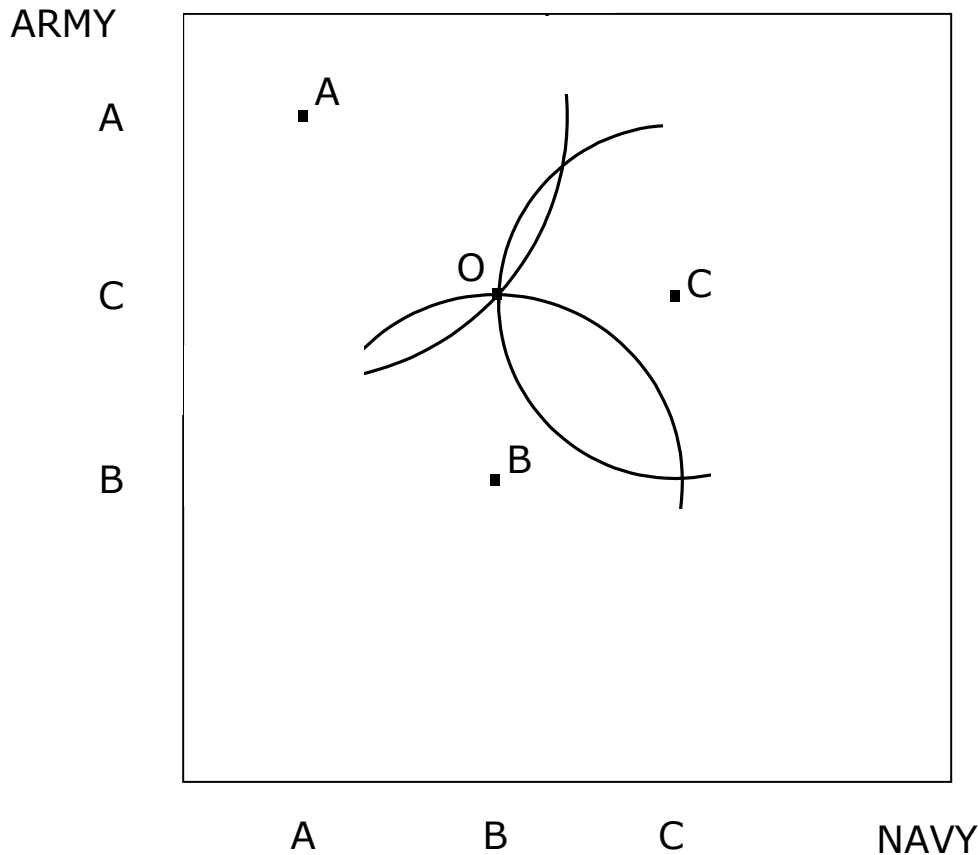
said that nobody had paid much attention to it recently except me, and in fact listed four places where I had mentioned it although I had not discussed it in any detail.

It's interesting that the work of Condorcet and the few other mathematicians including Dodson did not produce any significant public interest in the problem. This was, of course, the period in which European hereditary monarchies were in many cases replaced by democracies. It would appear that the mathematical arguments for the weakness of democracy were put into the memory hole. When the mathematical arguments for weakness of democracy were greatly reinforced by Arrow, Black and Newing, there was only a short upsurge in interest and then once again it went into the memory hole.

I presume that the absence of interest in these proofs is because they cast doubt on the desirability of democracy. Western intellectuals, particularly since the collapse of the Soviet Union in 1990, have been devoted to the idea of democracy. It is true that those who favored socialism, which in early days included Arrow, sometimes seemed to not notice that the most important socialist countries were dictatorships. But since 1990 that has not been true and not even the very great economic success of non-democratic China seems to have been held up as a contrary model. In any event at the moment Western intellectuals are not only enthusiastic about democracy, but most are in favor of a world trade system, even those who worry about "globalism".

The purpose of this essay is not to prove that the various attacks of a mathematical nature on democracy are false, but to indicate that in spite of these perfectly valid defects of a mathematical nature it reaches a not catastrophically bad conclusion. But the demonstration that the outcome, if not perfect, is not catastrophically bad, must be put off until the mathematical and geometric groundwork has been laid.

We may begin by demonstrating that the single peak preference curve, which Black and Newing were so proud of, and which was accepted by Arrow, is a broken reed. For this purpose I need a little geometry. Assume the government body is making decisions on the military and will have separate votes on the Navy appropriation and the Army. The Navy appropriation is shown on the horizontal axis below and locations of 3 people, A, B., and C are shown by the points on the horizontal axis. This is a single peak preference curve and B could anticipate winning because for any point to his right A would join with him and for any point to his left C would join with him.



There is however the Army appropriation and I place it on the vertical axis. Once again this is a single peak preference curve and in this case C has the median preference and hence C's optimum would prevail. If, however, we plot these preferences on the two-dimensional issue space we find the outcome of these two votes at O. We can also plot the voters preferences in this space and we find the three optimum at the labeled points. If we draw indifference curves around each of these optimum for the three voters, using circular curves for simplicity, we immediately observe a set of lozenge shaped areas. A motion for any point in any of these lozenges can get a two to one majority over the optimum selected by the median preference on the two axes. Even if separate votes on the Army and the Navy had passed, a motion to choose another joint point would be in order, and hence the cycle would return. We are now back to Black's original proof, that with the type of issue space we normally use to represent the real world, there is no majority voting optimum. Every point is dominated by other points. Even points in the pareto optimal area, which in this case is a triangle with its corners at each of the three individual optima, can lose a majority vote against points outside. Thus we are back in the situation before Black invented the single peak preference in order to save democratic voting.

There is however another argument that majority voting reaches a pretty good result. This was invented by me and published as “the general irrelevance of the general impossibility theorem”⁶.

Putting the matter in very general terms, I demonstrated that any point in the issue space is dominated by at least one point which is closer to the center until one gets very close to that center. In a small area near the center every point is dominated by at least one other point so that endless circling could occur. I argued that the rules of order rule out very small amendments and hence the voting would stop somewhere near the center of the cloud of optima in the issue space.

Plott and Demeyer in a famous article showed that actually reaching the middle point of the cloud of issue Optima was almost impossible, at the very least very improbable⁷.

This proof, like most other demonstrations that democracy does not produce a perfect result, seems to have vanished into the memory hole except in so far as it is thought to be of technical interest by occasional students.

For my proof I assumed that the cloud of optimum of the voters was normally distributed⁸, thus in a way, they were concentrated in a two-dimensional normal array. This is not necessary of course, and the end of the article I discussed briefly two other possible arrangements of the optima. Arrow in a generally approving letter to me pointed out that this was not mathematically strict. He was of course right but now we have somewhat more information on the point.

Reaching the approximate center would have required a long series of votes between alternatives that are very similar to each other. Further if we had an all-powerful all knowing chairman controlling vote order, we can get almost anywhere in the space⁹. Hinich and Munger have done empirical work which indicates that there is what I would call a ridge of the Optima running roughly speaking from left to right in the usual idea of the issue space.

I have doubts about this because I think it may represent the way we think about the problems rather than the way they are. Strictly speaking, for example, farmers everywhere want more money. Before World War I this desire was put on the right. It is

⁶ Tullock, Gordon, “The General Irrelevance of the General Impossibility Theorem” , Quarterly Journal of Economics, Vol LxxxI, May 1967, pp 256-70.

⁷ Plott, Charles & F. Demeyer “The Probability of a Cyclical Majority” *Econometrica*, 38, March 1970, pp. 345-54.

⁸ For elegance; I should say that the space I dealt with was a hyperplane cutting through the multidimensional issue space. The voters optima were points on the hyperplane closest to their optima in the multidimensional issue space.

⁹ Hinich, Melvin and Munger, Michael, *Analytical Politics*, Cambridge University Press, 1997.

now put on the left. I think this is more a statement about our feelings than about the real world. I suspect many of the other issues that we list as right or left could just as well be accurately plotted on a vertical to the left right order.

I have, however, come to the conclusion that my original proof is refuted by the real world. If we look at the proceedings in any legislative body we find that the final vote is normally between two alternatives that are quite distant from each other. Shall we or shall we not appropriate some particular amount of money for example. Even when they vote on quantities of appropriations they're frequently quite distant. Certainly neither of the two alternatives brought up on the floor would be eliminated by the rules of order as being trivial. Further they do not approach an ultimate Optima by a series of very small steps. Frequently all that happens on the floor of Congress is an up or down vote on any given issue.

Thus my previous proof, although logically compelling, does not fit the observed data. The final decision is made between alternatives which are quite far apart. The most recent major squabble in United States Senate concerned whether or not a sum of \$17 billion should be granted to the provisional Iraqi government or only loaned. That is not a small matter and was not treated as a small matter either in the debates in Congress or in the public debates in the newspapers and on TV. I, naturally, feel unhappy when a logically compelling and apparently perfect proof fails because the world doesn't comply.

A great deal of work in modern legislatures is done in committee and we do not usually have full reports of what goes on. Thus I cannot say that the proceedings of the committee do not continue until a point is reached for which no majority wants to make a significant major change. Although I do not have conclusive evidence I think that is not the case.

We thus find ourselves in a most peculiar situation. We appear to have mathematical proofs that democracy would not work, but it is functioning all over the world. Indeed at the moment we are at a high in democracy with almost half of the world's population living in democracies and with practically all intellectuals urging democracy very vigorously. The last is, as mentioned above, rather recent. Intellectuals during the time I was attending school and then meeting many of them professionally, tended very heavily to be socialists. This did not make them opposed to democracy, but it did make them somewhat tolerant of communist dictatorships. Since 1990 and the abandonment of communism by Russia this tolerance has disappeared.

How then can we explain the existence of apparently good proofs that democracy won't work with this widespread application and approval. As I mentioned above the disproofs of democracy have tended to be shoved into the memory hole and hence are not taken into consideration normally when forms of government are discussed. Still it is not true that democracy produces results of a catastrophic nature. I suspect there's no person who thinks that the outcome of the political processes is optimal or even close, but to repeat, they are not catastrophic.

But let me put this aside temporarily, and now turn to another problem raised by Arrow, the lack of independence of irrelevant alternatives. By long-standing custom this is normally illustrated with the use of the method invented by Borda, and a typical demonstration is shown herewith.

1	2	3	4	5
A	A	A	B	B
B	B	B	C	C
C	C	C	A	A

In the table 5 voters Mr. 1, 2, 3, 4, and 5 have preferences over three alternatives A, B, and C as shown. If we list use Borda's method, there will be one point to each of the bottom choices two for the second from bottom choices and three for the top. This gives Mr. B. 12 points, Mr. A 11 points and Mr. C. 7. Mr. B. then wins. If however C is not present and we give one point to each point in the bottom row, and two for each on the upper row, then A has eight points and B has seven, A wins. C, which has no chance of winning, is the irrelevant alternative, and its presence or absence determines who does win. That irrelevant alternatives have no effect on the outcome was one of Arrow's conditions. Without it his proof fails.

Let me, however, introduced a little science-fiction. We do not know now what will be discovered in the next hundred years. As we learn more about the brain it may be possible for us, with the aid of supercomputers, to determine how strong various people's feelings are. Probably the rating of the alternatives in the Borda method is to some extent an indication of the strength of people's feelings on the various alternatives. It is of course a very poor measure, but there's no reason to believe that we may not have, 100 years in the future, a very good measure.

If we had such a measure we might prefer not that alternative which has the highest number of first preferences but that alternative which maximizes the total satisfaction received by the voters. Thus if Mr. 5 had only a rather weak preference of alternative A over alternative B, we might wish to discount that preference somewhat. Maximizing total satisfaction, if we could do it, would seem to be better than simply selecting the alternative which is preferred by the most voters even if in some cases that preference is very weak.

As an example consider three voters, 1, 2, and 3 choosing among five alternatives, A, B, C, D, and E. Their preferences are as in the following table. We will also assume following Borda that they honestly represent their preferences when voting.

1	2	3
A	A	B
B	B	C
C	C	D
D	D	E
E	E	A

Using normal rules of order, A would get a two to one vote against each of the alternatives. Using Borda's method, however, B would win because of the apparent strong dislike of A by voter three. It's not obvious which outcome we would prefer and the difference is the existence of C, D, and E, what Arrow called, irrelevant alternatives which are discounted by the rules of order, but taken into account by Borda. To repeat, it's not obvious whether A or B is the optimum. A politician contemplating a future election would almost surely choose B.

I suppose a radical egalitarian might prefer to produce a relatively equal level of satisfaction over the society and with our supercomputers this would be possible. It is likely that the outcome of either of these two science-fiction methods would be superior to simple majority voting.

In other words, the undesirability of the lack of independence of irrelevant alternatives may be dependent on our inability to read people's minds, and that is not necessarily a permanent condition. It is the case that the Borda method gives us some crude information about how strongly people feel and hence has this advantage to counterbalance against its lack of independence of irrelevant alternatives.

We do not have the supercomputer but we do have an approximation, perhaps a very bad approximation. This is the politician. There's no doubt politicians attempt to make up bundles of measures in which one person's injury on one issue is at least partially counterbalanced by his gain on another. Politicians are generally very good at guessing what people think and I think they might make a fairly good approximation of our supercomputer. Presumably use of the Borda method would also be an approximation but a very, very bad one.

Thus it is not obvious that independence of irrelevant alternatives is something which we should hold up as an absolute criteria. I cannot think of any way of settling the problem definitely, but it is clear that we do not actually know enough to be sure that we should avoid all methods which, crudely, give some information on the strength of feelings of the voters.

Turning to the history of monarchies, a monarch would normally find himself imposing quite a large number of laws on his subjects. He might, like the politician, make efforts produce a bundle which offset a disadvantage to citizen A with an advantage somewhere else in the bundle. This would be to the monarch's advantage in reducing opposition and to the citizen's advantage as well. If you read history you find people classifying monarchs as good or bad kings. It may be the good kings try to do this and do it well, while the bad kings either do not try or are inept.

In order to offer my explanation of what actually happens in democracy, I would like to go back in history. Most human beings have lived under some kind of monarchy, either the temporary monarchy that we call dictatorships or the more permanent arrangement which has dominated most countries through history which has some kind of hereditary

succession¹⁰.

When the Emperor of Persia, and Persia was the first really large empire and was successful for a long time, wanted to make a decision on some point he assembled a collection of counselors, sat on his throne, and asked for advice. When he thought he had enough counsel he would say: "It is my will that,(for example). we attack Athens¹¹."

The advisers would then shout in chorus "It is the Kings will."

No doubt the Emperor realized that it was not clear he had an optimum decision. Further discussion, perhaps other counselors, might lead to an improvement. It might also, of course, lead to a worse decision. It would certainly take some of his time, which could be more pleasantly spent in other occupations. In a way he made a cost benefit decision putting the cost of further discussion against the possible benefit of improved actions.

As a brief digression here, he didn't come off too badly. The giant Persian empire was in almost continuous wars with minor powers on its outskirts. It didn't always win, but the long run the trend until Alexander was favorable. Indeed even Greece, not too long after Salamis, lived under the King's Peace.

If we look at other monarchies we see much the same procedure. The ruler, whether the King of France, Stalin, or the Pope solicits advice from a council and then makes up his mind. Sometimes hereditary rulers can be quite stupid. Louis the 16th was. He used to fall sleep at council meetings. Still this is the basic procedure used in monarchies and the results are, if not optimal, normally not catastrophic. Sometimes of course they are very bad indeed. I have recently been reading a history of the opening round of the hundred years war. Two separate Kings of France and a Dauphin, who was ruling because his father was in prison in London, led large armies to disastrous defeats. But it is not only monarchies that have defeats. We lost the Vietnamese war.

As a reader has no doubt realized, I think that voting bodies, like Kings, do not wish to put in massive amounts of time into decisions. Whether they consciously realize that they could improve their output by further debate is not obvious. Further is not obvious in general that if they did continue the debate, they would really improve. As a general rule more careful consideration improves decisions, but that is not always true. Further spending more time and energy in further consideration is a cost and it may be greater

¹⁰ see my Autocracy, Kluwer, for a fairly full discussion of this matter. I have recently written another study which is a chapter in my "Externality, Redistribution, and Rent Seeking" (forthcoming, Elgar)

¹¹ "Truth loving Persians seldom dwell upon the minor check suffered at Marathon In spite of bad luck and unfavorable weather all arms coordinated magnificently together."

Anonymous. Produced in England shortly after the Greek campaign in 1941.

than the improvement. The legislative outcome, then, would normally not be very close to the center of the cloud of individual optima, but well away from it. Accuracy is sacrificed in order to save time.

Thus in both the democratic and the monarchical cases the outcome is not the optimum and probably far from the optimum. The difference is that the optimum in one case is a benefit for the King and in the other case benefit for the voters. Most of us prefer the second optimum even if we realize the decision process is very, very, far from perfect. Thus we get what is probably a bad approximation of the “public good” rather than another bad approximation of the King’s good.

The voting process in democracy then does indeed have severe paradoxes and if it avoided them would, in any event, be the will of a group of people, the voters, who are neither very intelligent nor very well-informed. It's only when we contrast it with the alternatives which are in fact available that we can argue for it. This is a weak argument for what is over history a minor form of government, but which may be en route to becoming the dominant form for the immediate future. I wish that I had something better.