Perfect Cake-Cutting Procedures with Money

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Abstract

A cake-cutting procedure is *perfect* if it satisfies the properties of efficiency, envy-freeness, and equitability. For two players there is no perfect procedure, including cut-and-choose and Austin's procedure. But the addition of money makes perfection possible, as we demonstrate by describing a 2-player, 1-cut perfect procedure that induces each player to be truthful in order to maximize the minimum portion of cake it can guarantee for itself.

For $n > 2$ players, an envy-free procedure can be rendered equitable with the addition of money, but not necessarily efficient if it uses more than n - 1 cuts (the minimal number). Although there is a minimal 3-player, 2-cut envy-free procedure, no 4-player, 3-cut envy-free procedure is known to exist.

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1. Introduction

By *perfect* we mean a cake-cutting procedure that satisfies three desirable properties—efficiency, envy-freeness, and equitability—which we will describe and illustrate shortly. This ideal, as Jones (2002) showed, can in principle be achieved for n = 2 people with one cut. However, no procedure, or set of rules, is known that gives two people an incentive to make the perfect cut.

In this paper, we show how two people, whom we henceforth call players A and B, can be motivated to make such a cut, but only when money is introduced. However, the perfect cut will not necessarily yield an efficient division: It is possible that both players can be made better off if A, say, compensates B for A's getting relatively more cake. We say "relatively more," because we insist that each player gets at least half the cake, as it values the cake, not just receive money instead of cake.

Thus, we preclude A from buying the entire cake and compensating B for getting none. While this might be an efficient solution if A values the cake highly and B does not, we henceforth assume that the players value the cake equally. This common value to the players might be the amount that the cake would sell for in the marketplace. We also assume the cake is a heterogeneous good, so A and B may have different private values for its parts.

We begin by describing two 2-person cake-cutting procedures, one well known and the other not, that satisfy two of the three properties, though different ones. We then describe a perfect 2-person procedure that uses money. Each player gets exactly 1/2 the

cake, as each values it; if any cake remains ("surplus"), the players get at least 1/2 of this portion, either as cake or its money equivalent.

If there are $n > 2$ players, envy-freeness (each player gets at least as much as the other players, so it does not envy anybody else) and equitability (all players value the portions they receive—cake and possibly money—the same) can be achieved with the addition of money, but not necessarily efficiency if more than $n - 1$ cuts (the minimal number) are required. While there exists a simple 3-person, 2-cut envy-free procedure (to be described), it is an open question whether there exists a 4-person, 3-cut envy-free procedure that, with the addition of money, yields a perfect division.

We assume throughout that the players value the cake along a line that ranges from $x = 0$ to $x = 1$. More specifically, we postulate that the players have continuous value functions, $v_A(x)$ and $v_B(x)$, where $v_A(x) > 0$ and $v_B(x) > 0$ for all x over [0, 1]. Analogous to probability density functions, or pdfs, we assume the total valuations of the players—the areas under $v_A(x)$ and $v_B(x)$—are 1. We also assume that only parallel, vertical cuts, perpendicular to the horizontal x-axis, are made, which we will illustrate later.

We postulate that the goal of each player is to maximize the value of the minimum-size piece that it can guarantee for itself, regardless of what the other player(s) do. When we bring money into the picture, we assume that it can be calibrated with cake, so different amounts of cake have their money equivalent for the players. In this situation, players seek to maximize the minimum amount of cake, and possibly money, that they receive. Put more succinctly, they choose *maximin strategies* that yield them *maximin values* of cake and money.

We show that the maximin strategies that give the players these guarantees require them to be truthful about their valuations of cake under our procedures. If they try to gain more by misrepresenting their value functions to a referee, they lose these guarantees. In the subsequent analysis, we assume that the players are *risk-averse*: They never choose strategies that might yield them more valuable pieces of cake or more money if they entail the possibility of giving them less than their maximin values.

2. Cut-and-Choose

The well-known cake-cutting procedure, "I cut, you choose," or cut-and-choose, goes back at least to the Hebrew Bible (Brams and Taylor, 1999, p. 53). It satisfies two of the three desirable properties (envy-freeness and efficiency).

Under cut-and-choose, one player cuts the cake into two portions, and the other player chooses one. To illustrate, assume a cake is vanilla over [0, 1/2] and chocolate over (1/2, 1]. Suppose the cutter, player A, values the left half (vanilla) twice as much as the right half (chocolate); this implies that $v_A(x) = 4/3$ on [0, 1/2] and $v_A(x) = 2/3$ on (1/2, 1].

We next show that A can ensure an envy-free division of the cake.

> A division is **envy-free** if and only if each player thinks it receives at least a tied-for-largest portion, so it does not envy another player.

To guarantee envy-freeness in the case of two players, A should cut the cake at some point x so that the value of the portion to the left of x is equal to the value of the portion to the right. The two portions will be equal when A's valuation of the cake between 0 and x is equal to the sum of its valuations between x and 1/2 and 1/2 and 1:

$$(4/3)(x - 0) = (4/3)(1/2 - x) + (2/3)(1 - 1/2),$$

which yields $x = 3/8$. In general, the only way that A, as the cutter, can ensure itself of getting half the cake is to give B the choice between two portions that A values at 1/2 each.

Besides envy-freeness, cut-and-choose satisfies one other desirable property, efficiency (also called Pareto-optimality).

> A division is **efficient** if and only if there is no other allocation that is better for one player and at least as good for all others.

In the case of two players, the more value one player receives the less value the other player receives. Hence, one player cannot benefit from a different cut without hurting the other.

Cut-and-choose does not satisfy the third desirable property, equitability.

> A division is **equitable** if and only if each player's valuation of the portion that it receives is the same as every other player's valuation of the portion that it receives.

To show that cut-and choose does not satisfy equitability, assume B values vanilla and chocolate equally. Thus, when A cuts the cake at $x = 3/8$, B will prefer the right portion, which it values at 5/8, and consequently will choose it. Leaving the left portion to A, B does better in its eyes (5/8) than A does in its eyes (1/2), rending cut-and-choose inequitable.

If the roles of A and B as cutter and chooser are reversed, the division will remain inequitable. In this case, B will cut the cake at $x = 1/2$. A, by choosing the left half (all

vanilla), will get 2/3 of its valuation, whereas B, receiving the right half, will get only 1/2 of its valuation.

3. Austin's Procedure

Austin (1982) proposed a "moving-knife" version of cut-and-choose, which we describe by picturing the cake as a rectangle in Figure 1 (Brams, Taylor, and Zwicker, 1995; Brams and Taylor, 1996, pp. 22-29). A referee slowly moves a knife from left to right across the cake so that, at every point along the horizontal axis, it remains parallel to its starting position at the left edge.

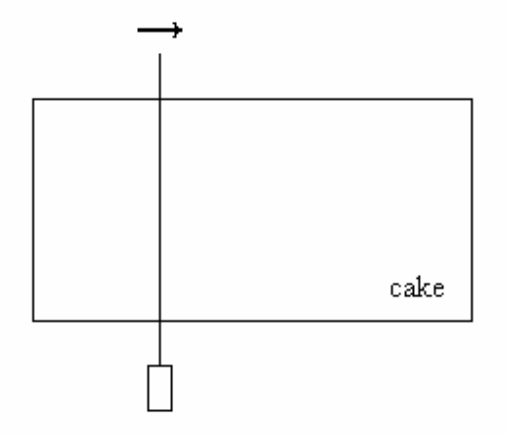


Figure 1. One Moving Knife

At any point, either player can call stop. When this happens, a *second* knife is placed at the left edge of the cake (see left side of Figure 2). The player that called stop then moves *both* knives across the cake in parallel fashion (see right side of Figure 2).

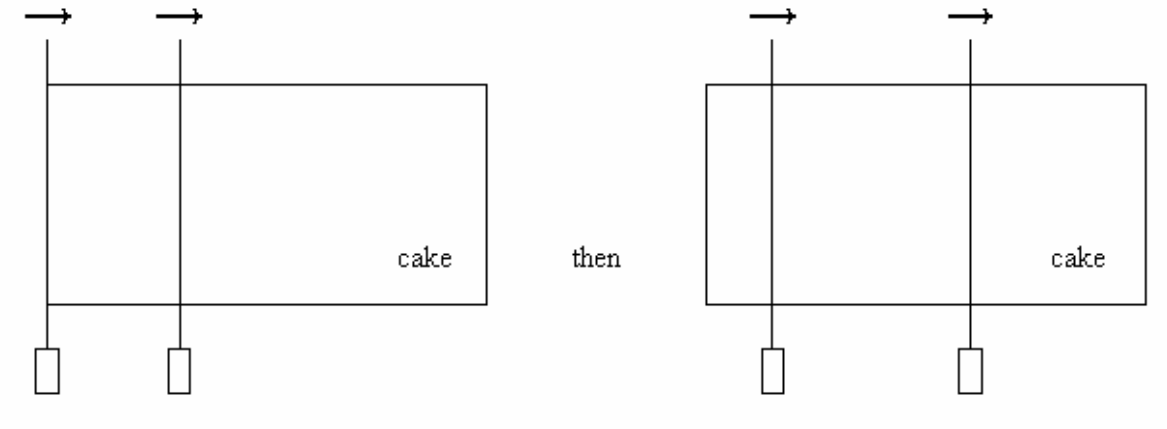


Figure 2. Two Simultaneously Moving Knives

Assume A is the first player to call stop and, therefore, is the one to move the two parallel knives across the cake. We require that when A's right knife arrives at the right-hand edge of the cake, its left knife lines up with the position that the first knife was in at the moment when A first called stop (shown on the left side of Figure 2).

While the two knives are moving, B can call stop at any time. After calling stop, one player, chosen at random, gets the portion between A's two knives (middle piece), and the other player gets the two outside pieces (end pieces). We assume that a player's valuation of the two end pieces is the sum of its valuation of each end piece, so its total valuation of these pieces is not affected by the fact that they are disconnected.

A little reflection shows that the only way that each player can guarantee that it gets 50% of its value of the cake is if the player that calls stop first (A) does so (i) when the first knife cuts the cake 50-50 in its eyes and (ii) it subsequently keeps exactly 50% between the two parallel knives as it moves them across the cake. By the same token, the only way the other player (B) can guarantee itself 50% is to call stop when the middle piece is exactly 50%, and hence the two end pieces sum to 50% as well.

What guarantees that there will be a point when B thinks the middle piece is exactly 50%? Observe that at the instant when the two knives start moving (left side of Figure 2), B thinks the middle piece is less than 50% (otherwise B would have called stop before

A). At the point when the two knives stop moving on the right-hand side of the cake (assuming B does not call stop earlier), the piece between the knives is the complement of what it was when the knives started moving.

Consequently, B thinks the middle piece, now on the right-hand side, is strictly greater than 50%. Assuming continuity in both the composition of the cake and the players' preferences for it, there must, by the intermediate-value theorem, be a point where B thinks the value of the middle piece is exactly 50%. If the choice of which player gets the middle piece and which player gets the end pieces is random, the players can each guarantee themselves 50%—thereby satisfying the maximin assumption—only by being truthful (Brams and Taylor, 1996, pp. 26-27).

Because both players get exactly 50%, Austin's procedure satisfies envy-freeness and equitability. However, it does not satisfy efficiency. To see this, let us apply Austin's procedure to our earlier example, in which A, to ensure envy-freeness, stops the first knife at its 50-50 point of $x = 3/8$.

Placing a second knife at $x = 0$, A then moves the two knives in parallel until B calls stop, which will occur when the position of the left knife is $x = 1/4$ and the position of the right knife is $x = 3/4$. Because each player gets 1/2 the vanilla and 1/2 the chocolate—whether it gets the middle piece or the two end pieces—each player gets exactly 50% of its valuation.

Recall that under cut-and-choose, B gets 5/8 when A, as the cutter, gets exactly 1/2. Likewise, when B is the cutter and gets exactly 1/2, A gets 2/3. Compared with Austin's procedure, therefore, one player does better, and the other player the same, proving that Austin's procedure is inefficient.

Manifestly, by limiting both players to exactly 1/2, Austin's procedure does not exploit the win-win potential of cake division when the players have different preferences for different parts of the cake. This is so even though Austin's procedure uses two cuts,

which ordinarily would enable the players to do better than a 1-cut procedure like cut-and-choose.

4. A New 2-Player, 1-Cut Procedure

We have shown that neither cut-and-choose nor Austin's procedure satisfies the three properties of a perfect procedure. Cut-and-choose is not equitable, and Austin's procedure is not efficient. Although Jones (2002) shows how to find a cutpoint x such that cutting the cake at x yields a perfect division, the players' truthful revelation of their value functions may not be a maximin strategy.

By allowing for the possibility, but not requiring, one player to make a monetary payment to the other, we introduce a procedure that induces the players to be truthful. Like all 2-person procedures that use only one cut, this procedure is efficient—no other 1-cut division of the cake, without money, can improve the lot of both players if they each attach positive value to all points of the cake. But unlike cut-and-choose and Austin's procedure, the new procedure, in which money may be used, satisfies a type of equitability (we will distinguish different types later) and efficiency as well as envy-freeness. Here are its rules, which we will refer to as *steps*:

1. Independently, A and B report their value functions, $f_A(x)$ and $f_B(x)$, over the cake, [0, 1], to a referee. These functions may be different from the players' true value functions, $v_A(x)$ and $v_B(x)$.

2. The referee determines the 50-50 points, a and b, of A and B—that is, the points on [0, 1] such that each player reports that half the cake, as it values it, lies to the left and half to the right (these points are analogous to the median points of pdfs).

3. If a and b coincide, the cake is cut at a/b. One player is randomly assigned the piece to the left of cutpoint a/b, and the other player the piece to the right. The procedure ends.

4. Assume that a is to the left of b, as illustrated below (ignore x for now):

0----------*a*-----*x*----------*b*----------1.

Then A is assigned the portion [0, *a*], and B the portion [*b*, 1], which each player values at 1/2 according to its reported value function. (If *a* is to the right of B, A is assigned the portion [*a*, 1], and B the portion [0, *b*], with appropriate changes made for the players in steps 5 and 6 below.)

5. Assume that A and B attach values, $S_A = \int_a^b f_A(x)dx$ and $S_B = \int_a^b f_B(x)dx$ (for *surplus*), to the cake in the interval (*a*, *b*). If $S_A > S_B$, A is assigned the surplus, and vice versa if $S_B > S_A$; if $S_A = S_B$, the surplus is assigned, at random, to a player. The player assigned the surplus makes a monetary payment to the other player such that both players receive the same *proportion* of their valuations of the cake in (*a*, *b*). Both the assignment of the surplus and the monetary payment are temporary.

6. The referee determines if the assignment and monetary payment will be made permanent. This depends on whether or not the players' valuations of the assignment/payment are "Pareto-dominant." An assignment/payment *Pareto-dominates* an equitable division at the perfect cutpoint *x*, giving (*a*, *x*] to A and (*x*, *b*) to B, if at least one player prefers the assignment/payment and the other player finds it at least as good:

(i) *Assignment/payment Pareto-dominant.* If A was temporarily assigned the surplus in step 5, the cake is cut at *b*; A gets [0, *b*), and B gets [*b*, 1] plus a monetary payment. If B was assigned the surplus in step 5, the cake is cut at *a*; B gets (*a*, 1], and A gets [0, *a*] plus a monetary payment.

(ii) *Assignment/payment not Pareto-dominant*. The cake is cut at the perfect cutpoint *x*; A gets [0, *x*] and B gets (*x*, 1].[1]

The procedure ends.

[1] We will illustrate how to find *x* when we return to our example.

To illustrate these steps, consider the example introduced in section 2, where

$v_A(x) = 4/3$ on $[0, 1/2]$ and $v_A(x) = 2/3$ on $(1/2, 1]$; $v_B(x) = 1$ on $[0, 1]$.

Assume that the players are truthful, so $f_A(x) = v_A(x)$ and $f_B(x) = v_B(x)$. (We will assess later whether truthfulness is a maximin strategy for the players.) Then the surpluses as given in step 5 for the players are

$$S_A = \int_{a=3/8}^{b=1/2} (4/3)dx = (4/3)[1/2 - 3/8] = 1/6.$$

$$S_B = \int_{a=3/8}^{b=1/2} (1)dx = (1)[1/2 - 3/8] = 1/8.$$

Because $S_A > S_B$, A is assigned the surplus and makes a monetary payment to B such that both players receive the same proportion of their valuations of the cake in (a, b). Let y be the payment that A makes to B, so A retains $1/6 - y$. The proportions of the cake in (a, b) that the players receive will be equal when $(1/6 - y)/(1/6) = y/(1/8)$, which yields $y = 1/14$.

After A makes a monetary payment of 1/14 to B, A will value its cake allocation, $[0, b)$, which includes the surplus, at $1/2 + (1/6 - 1/14) = 25/42$. B will value its cake allocation, $[b, 1]$, and money at $1/2 + 1/14 = 24/42 = 4/7$.

We now compare these valuations with what a perfect cut by the referee at x would give, as described in step 6 of the procedure. From the value functions given earlier, we can determine the *cumulative* value functions of the players, or what A would receive to the left of a cut at x and what B would receive to the right of this cut:

$V_A(x) = 4x/3$ on $[0, 1/2]$ and $V_A(x) = 2/3 + 2x/3$ on $(1/2, 1]$; $V_B(x) = x$ on $[0, 1]$.

The perfect cut will be in $[0, 1/2]$, because A values this interval at more than 1/2 and B at 1/2.

Jones (2002) showed that the point x that equalizes the valuations of A and B exists and occurs when

$$V_A(x) = 1 - V_B(x)$$
$$4x/3 = 1 - x,$$

which yields $x = 3/7$. Thus, A receives the cake in [0, 3/7], which it values at 4/7, and B receives the cake in (3/7, 1], which it also values at 4/7.

Comparing these valuations (4/7 each) with what the players receive when A gets the surplus in (a, b) and makes a monetary payment to B (25/42 to A, 4/7 to B), we see that A does better from receiving the surplus (because 25/42 > 4/7), whereas B does the same (4/7), so the assignment/payment option Pareto-dominates the perfect-cut option. According to step 6, therefore, A gets the surplus and makes a monetary payment to B, which gives both players equitable portions of the surplus.

To continue our example, assume that both players value the entire cake at \$24. Then A's valuation of the surplus in (a, b) is \$24/6 = \$4, and B's valuation of this surplus is \$24/8 = \$3. If A transfers \$24/14 ≈ \$1.71 to B, each player will obtain 1.71/3 ≈ 2.29/4 ≈ .57 proportion of its valuation of the surplus.

This division gives what we call *surplus equitability*, but it is not the only way to divide the surplus. In section 5 we will describe three other ways of doing so and offer our evaluation of these alternatives. First, however, we prove that the procedure we have just described and illustrated gives a maximin outcome for risk-averse players.

Theorem 1. *The procedure described by steps 1 – 6 guarantees A and B exactly 1/2 the cake as each values it, and at least 1/2 the surplus—in money or cake, whichever is greater—if and only if they are truthful in reporting their value functions in step 1.*

Proof. We start with the "if" part. If the players are truthful, and *a* is to the left of *b* (as shown in the earlier figure), then [0, *a*] and [*b*, 1] give A and B, respectively, exactly 1/2 their valuations of the cake.

As for the surplus in (*a*, *b*), the minimal equitable amount that the players receive if a monetary payment is made is 1/2 their valuations of the surplus. This minimum occurs when the players' valuations of the surplus are exactly the same, so the player that gets (*a*, *b*) pays 1/2 its valuation to the other player. If the players' valuations are not the same, both players receive an equitable proportion greater than 1/2, because the player with the higher valuation needs to pay less than 1/2 its valuation to the other player to achieve equitability. If the assignment/payment option is not Pareto-dominant, then both players do at least as well with the perfect-cut option, giving them the same amount and guaranteeing both players at least 1/2 the surplus.

For the "only if" part, assume that one player (say, A) is not truthful in announcing its value function. There are two cases to consider:

1. A's misrepresentation causes *a* to crisscross *b*, as illustrated by the position of *a*' below,

0----------*a*---*b*--*a*'----------1.

Then A will obtain [*a*', 1], and get some less-than-complete portion of (*b*, *a*'), which is less than 1/2 the cake for A.

2. A is truthful about its 50-50 point, *a*, but it misrepresents its valuation of (*a*, *b*). Then there are two cases to consider in determining what amount of the surplus A will obtain:

(i) A indicates that (*a*, *b*) is worth *more* than its true value;

(ii) A indicates that (*a*, *b*) is worth *less* than its true value.

In case (i), suppose that it appears that $S_A > S_B$. Then A will get all the cake in (a, b) and have to pay B some amount for it. Assume this assignment/payment option Pareto-dominates the perfect-cut option and so, according to step 6, is chosen. Then if S_B is "close to" S_A, A will pay B almost 1/2 of S_A. Because of A's overvaluation of (a, b), however, this amount may be more than 1/2 A's true valuation of (a, b), leaving A with less than 1/2 its true valuation (even though it would appear to be more than 1/2).

In case (ii), suppose that it appears that $S_B > S_A$. Then B will get all the cake in (a, b) and have to pay A some amount for it. Because of A's undervaluation of (a, b), the amount that it receives from B may be less than 1/2 A's true valuation of (a, b) (even though it would appear to be more than 1/2).

In summary, we have shown that truthfulness guarantees the players exactly 1/2 their valuations in cake, and at least 1/2 the surplus, whereas a player's misrepresentation may lead to its receiving less of the cake or the surplus. Q.E.D.

We know of no 2-person procedure that, without money, gives players these guarantees. One reason is that the players may not have an incentive to be truthful about their value functions, especially near their 50-50 points, if undervaluing near these points can get them a larger portion. For example, if A reports, untruthfully, that it places little value on the cake just to the right of a in the diagram given in step 4 of the procedure, the perfect cut at x will be shifted rightward. Thereby A will get more cake—but only if the cake is cut at x.

Our procedure, however, precludes the cake from being cut at x if giving the surplus to one player, and having it make a monetary payment to the other, Pareto-dominates the perfect-cut option, as our earlier example illustrated. If A's undervaluing of (a, b) leads to the referee's choosing this option and awarding the surplus to B, A will do worse than had it been truthful about its valuation of (a, b).

Even if the players are truthful, an equitable division of the cake at x may not give each player at least 1/2 the surplus. This will occur when the players place more value on the cake near their opponent's 50-50 point than their own. In this situation, both players will do better under the assignment/payment option.

It is worth noting that our procedure can be adapted to dividing "bads," such as the time spent doing chores, as well as goods (Peterson and Su, 2002). In the case of bads, the incentives of the players will be reversed: Being truthful will enable them to minimize the maximum amounts of the bads they receive, giving them minimax rather than maximin portions.

More specifically, if it is bads that are being divided, let A get [b, 1] rather than [0, a], and let B get [0, a] rather than [b, 1]. Thereby each player gets no more than 1/2 the bads. As for the division of (a, b), one option would be an equitable division of the bads at point x, ensuring that neither player, even after the inclusion of the surplus, gets more than 1/2 the bads. The other option would be that the player that places more (negative) value on the surplus would pay its opponent to take the surplus from it in proportion to the players' (negative) valuations. The referee would choose the second option if it was Pareto-dominant—less harmful to one player and not more harmful to the other.

5. Possible Variations

The notion of "surplus equitability" that we proposed in section 4 generates an equitable division of only (a, b), not the entire cake. To accomplish the latter, the player that values (a, b) more would have to transfer 1/2 its valuation to the player that values it less. Thus in our earlier example in which A's valuation of (a, b) was \$4 and B's valuation was \$3, A would get (a, b) and pay \$2 to B. Thereby B would get 66.7% (2/3) of its valuation of (a, b), whereas A would get 50.0% (2/4) of its valuation.

Observe that both players get the same valuations of the *entire* cake, based on this new notion of equitability, which we call *overall equitability*. In monetary terms, A gets

\$12 from [0, a], and B gets \$12 from [b, 1]. In addition to these 50% cake portions, A gets \$2 worth of cake (after it makes a monetary payment to B), and B gets \$2 in money, giving A and B each \$14, or 58.3% of their valuations of the entire cake. This compares with allocations of \$12 + \$2.29 = \$14.29 (59.5%) for A, and \$12.00 + \$1.71 = \$13.71 (57.1%) for B, under surplus equitability (see section 4).

Whether equitability is surplus or overall, the sum of the valuations that the players receive is the same (116.7% in our example). Beyond the cake portions, [0, a] and [b, 1], that A and B receive, which are worth 50% to each, or 100% to both, the players get portions of the surplus that sum to the valuation of the player that values it more (A in our example).

Not only do surplus equitability and overall equitability provide different ways of dividing the surplus, but other ways of dividing the surplus fairly have also been proposed. We list below four proposed divisions in the order in which they favor A. In each case, we give

(i) the monetary division in our earlier example, in which A valued the surplus at at \$4 and B at \$3;

(ii) formulas for the divisions and their rationales, in which α is A's valuation and β is B's valuation of the surplus (we assume $\alpha > \beta$).

1. *Vickrey equitability*: \$2.50 to A, \$1.50 to B.

A pays 1/2 of B's valuation to B, so A gets $\alpha - \beta/2$ and B gets $\beta/2$. This division takes its name from a Vickrey (1961) auction, wherein the high bidder wins the good but pays only the second-highest bid. However, the surplus in (a, b) is not won by one player but shared: A wins the surplus but pays 1/2 of B's valuation of it. The justification for Vickrey equitability is that the player that values the surplus more (winner in an auction) should have to compensate the other player (loser) for only 1/2 the loser's valuation of the surplus.

2. *Surplus equitability*: \$2.29 to A, \$1.71 to B.

A gets $[\alpha/(\alpha + \beta)](\alpha)$ and B gets $[\beta/(\alpha + \beta)](\alpha)$, giving each player an allocation proportional to its valuation of the surplus. Its justification is that the player that values the surplus more should have to compensate the loser in proportion to its valuation of the surplus.

3. *Knaster equitability*: \$2.25 to A, \$1.75 to B.

A gets $(3\alpha - \beta)/4$ and B gets $(\alpha + \beta)/4$, which is a division that was first proposed by Knaster (1948). Its justification is that each player should get 1/2 its valuation ($\alpha/2$ for A; $\beta/2$ for B), with the remainder ($\alpha - [\alpha + \beta]/2 = [\alpha - \beta]/2$) split equally between the two players (Brams and Taylor, 1996, pp. 52-57).

4. *Overall equitability*: \$2.00 to A, \$2.00 to B.

A pays 1/2 of its allocation to B, so A and B each get $\alpha/2$. This is the mirror image of Vickrey equitability, whereby A's valuation rather than B's is divided equally between the players. Its justification is that equitability should apply to the entire cake, so each player should receive the same valuation for its portion.

In our view, a player's compensation for not receiving the surplus (a, b) should be responsive to *both* players' valuations of the surplus, which is not the case for either (1), in which only B's valuation (β) is divided, or (4), in which only A's valuation (α) is divided. When only one player's valuation is divided, the player that believes it plays this role will have a strong incentive to misrepresent its preferences to try to do better. Specifically,

- under (1), B will have an incentive to increase its valuation in order to raise the payment that it receives from A;
- under (4), A will have an incentive to decrease its valuation in order to lower the payment that it makes to B.

The compensation given to B by (2) and (3), which give intermediate allocations, makes them less easy to manipulate, because they depend on the valuations of both players. Between (2) and (3), we favor (2) on equitability grounds—it gives the players the same *proportions* of their valuations of the surplus in (*a*, *b*), as we showed earlier. By contrast, (3), under which each player initially receives 1/2 its valuation ($2.00 to A, $1.50 to B)—with the remainder ($.50) divided equally between the players—favors the player that values the surplus less (B in our example) by giving it a disproportionaly large share.

In fact, if B's valuation were $1.00 (instead of $2.00) in our example, but A's stayed the same ($4.00), B would be paid $1.25, which is more than B values the surplus. Is it fair that a player be paid more than what it thinks something is worth?

Instead of having the players report their valuations of the cake, one could ask them to submit bids for the entire cake and then have the winner compensate the loser according to one of the equitability concepts. However, if one player is wealthy and the other player is not, the cake's $24 price tag for the wealthy player might be a drop in the bucket for it. This would enable the wealthy player easily to outbid its non-wealthy opponent and obtain the entire cake, even though, in proportion to its wealth, the wealthy player might value the cake less.

This does not seem fair if we want to guarantee that the non-wealthy player gets at least some portion of the cake. We could do this by restricting bidding to the surplus in (*a*, *b*), which would ensure that the non-wealthy player always gets at least 1/2 the cake, as it values it, so we would remain in the realm of cake-cutting.[2] But in this paper we rule out bidding—in favor of the players' submitting value functions—to eliminate wealth distinctions.

[2] Note that the non-wealthy player might actually derive greater benefit from a monetary payment for the entire cake rather than being paid for only the surplus. Thus, whether a fair-division procedure is to be recommended will depend on whether players prefer having fair shares of cake or acquiring money instead.

We showed in section 4 that a perfect cut of the cake at $x = 3/7$ gives each player 4/7 of \$24.00, or \$13.71 each, in our example. Because surplus equitability in this example gives \$14.29 to A and \$13.71 to B, it Pareto-dominates the perfect cut, as does Knaster equitability and overall equitability. Only Vickrey equitability, which gives less to B (\$13.50), does not Pareto-dominate the perfect cut.

The 2-person cake-cutting procedure we described in section 4 for surplus equitability also works for any of the alternative notions of equitability we have discussed. Each may or may not require the use of money, depending on whether the assignment/payment option does or does not Pareto-dominate the perfect-cut option.

Perhaps the most straightforward perfect cake-cutting procedure with money starts from an envy-free division and then asks each player to contribute money to a pot, above and beyond its minimum 50% valuation, that will subsequently be divided between the players. To illustrate this procedure, consider cut-and-choose, in which we showed in section 2 that if A was the cutter in our example, it would cut the cake at $x = 3/8$; B would then choose the right piece, which it values at 5/8, and A would get the left piece, which it values at 1/2.

Assume that the amount of money that each player contributes to the pot is the difference between its valuation of its piece and 1/2 its valuation of the entire cake. Because A's valuation of its piece is exactly 1/2, or \$12, it contributes nothing. On the other hand, B's valuation of its piece is 1/8 above 1/2, or \$24/8 = \$3.00, so it contributes this amount to the pot. If this amount is divided equally between the players, each ends up valuing its portion at \$13.50, making the division overall equitable as well as envy-free.

Can the players in our example do better? If B is the cutter, we showed in section 2 that A would get 2/3 and B would get 1/2. This time A contributes 1/6, or \$24/6 = \$4, to the pot, which, when divided equally, makes each player's valuation of its portion \$14, duplicating the overall-equitability result of our earlier procedure.

This is no accident. Our earlier procedure asked which player puts a higher valuation on the surplus in (a, b). This was A in our example, which valued the surplus at \$4, compared with B, which valued it at \$3; as our example illustrates, the award of the surplus to the player that values it more is necessary to ensure efficiency. Thereby cut-and-choose, amended so that the chooser is the player that values the surplus in (a, b) more—and makes a monetary payment to the cutter—gives overall equitability if the chooser's valuation is divided equally between the players.

But any of the other notions of equitability could as well be used—in particular, surplus equitability—rendering cut-and-choose with money equivalent to our earlier procedure, absent the perfect-cut option. This option, however, is worth retaining, especially when a perfect cut Pareto-dominates surplus equitability. Although the opposite was true in our earlier example, the perfect cut may Pareto-dominate the assignment/payment outcome, as illustrated in the following example.

Example 1. Perfect Cut Pareto-Dominates Assignment/Payment Outcome

Assume the players submit the following value functions,

$f_A(x) = 2 - 2x$ and $f_B(x) = 2x$ on [0, 1],

as pictured in Figure 3. It is not hard to show that the assignment/payment option, whether the cut is made at $a = 1 - \sqrt{2}/2 \approx 0.293$ or $b = \sqrt{2}/2 \approx 0.707$, gives each player $0.5 + (\sqrt{2} - 1)/2 = \sqrt{2}/2 \approx 0.707$, whereas the perfect cut at $x = 1/2$ gives each player 0.75.

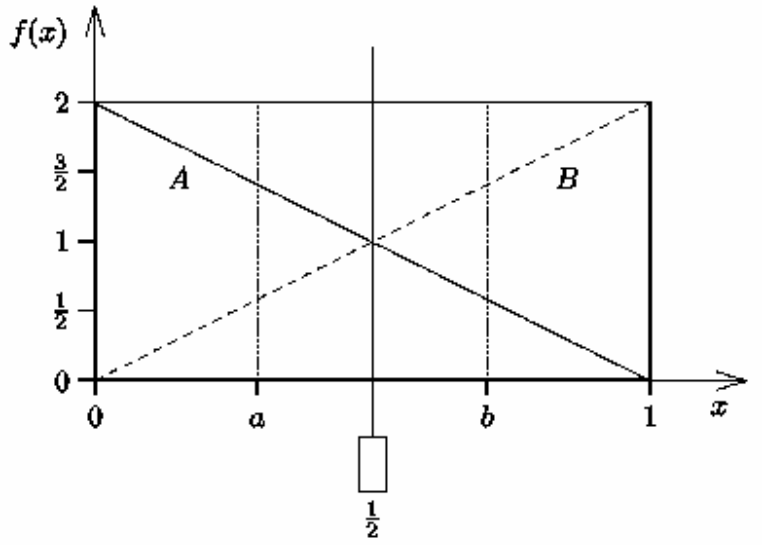


Figure 3. Perfect Cut Pareto-Dominates Assignment/Payment Option

Outside the realm of cake-cutting, Raith (2000) provides a good diagrammatic exposition of different notions of equitability. Meertens, Potters, and Reijnierse (2002) show that envy-freeness and efficiency may be incompatible in certain economies. Chisholm (2000) shows how envy-freeness, which all the equitability concepts satisfy in the 2-person case, can be extended to the *n*-person case under certain conditions that we will describe in section 6.

6. Perfectness for *n* Players?

It turns out that it is not always possible for three players, using only two cuts, to divide a cake into three envy-free and equitable portions, as demonstrated in the following example.

Example 2. Impossibility of Perfect Cuts for 3 Players

Assume that A and B have (truthful) piecewise linear value functions that are symmetric and V-shaped,

$$v_A(x) = \begin{cases} -4x + 2 \text{ for } x \in [0,1/2] \\ 4x - 2 \text{ for } x \in (1/2,1] \end{cases}$$

$$v_B(x) = \begin{cases} -2x + 3/2 \text{ for } x \in [0,1/2] \\ 2x - 1/2 \text{ for } x \in (1/2,1] \end{cases}.$$

Whereas both functions have maxima at $x = 0$ and $x = 1$ and a minimum at $x = 1/2$, A's function is "steeper" (higher maximum, lower minimum) than B's, as illustrated in Figure 4. In addition, suppose that a third player, C, has a uniform value function,

$$v_C(x) = 1 \text{ for } x \in [0,1].$$

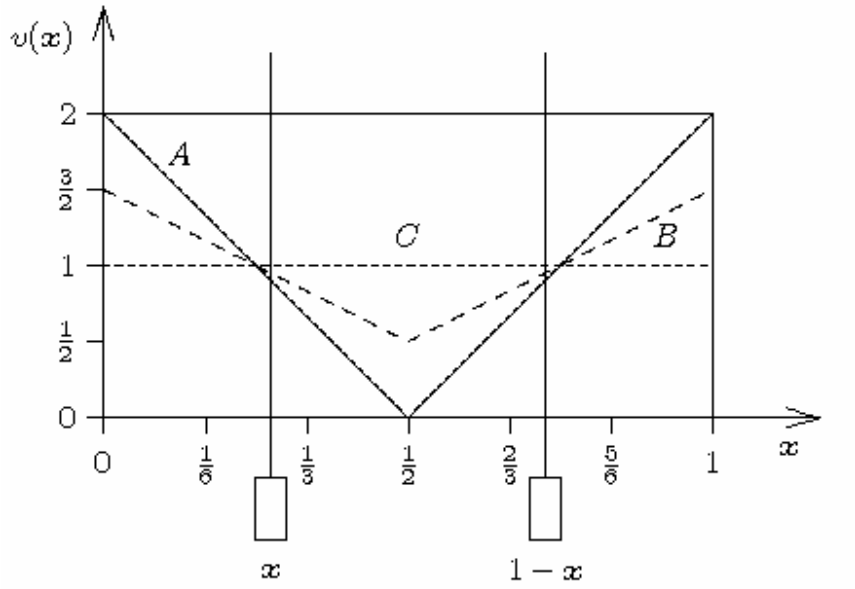


Figure 4. Impossibility of Perfect Cuts for 3 Players

In this situation, every envy-free allocation of the cake will be one in which A gets the portion to the left of x, B the portion to the right of $1 - x$ (A and B could be interchanged), and C the portion in the middle. If the horizontal lengths of A and B's portions are not the same (i.e., x), the player whose portion is shorter in length will envy the player whose portion is longer. But such an envy-free allocation will not be

equitable, because A will receive a bigger portion in its eyes than B receives in its eyes, violating equitability. Thus, an envy-free allocation cannot be equitable in this example, nor an equitable allocation envy-free, though both these allocations will be efficient with respect to parallel, vertical cuts.[3]

Nevertheless, we will show in this section how an *n*-person envy-free procedure can be made equitable with money. If such a procedure uses only $n - 1$ parallel, vertical cuts, it will also be efficient with respect to all procedures that use this minimum number (Gale, 1993; Brams and Taylor, 1996, pp. 150-151).

Two envy-free procedures have been found for 3-person, 2-cut cake division. Whereas one of the envy-free procedures requires four simultaneously moving knives (Stromquist, 1980), the other requires only two simultaneously moving knives (Barbanel and Brams, forthcoming).

To illustrate how money can be grafted onto an *n*-person procedure to create equitability, we describe the simpler procedure of Barbanel and Brams. It assumes virtual cuts, or what Shishido and Zeng (1999) call "marks," can be made on the line segment defining the cake. These marks may subsequently be changed by another player before real cuts are made.

In the description that follows, we describe an "instruction" that the players are given in the beginning. If it is followed, an envy-free allocation results; if it is not followed, then a player may do worse, violating the maximin goal we assume of players.

Barbanel-Brams 3-person, 2-cut procedure. A referee slowly moves a knife from left to right across a cake (see Figure 1). The players are instructed to call stop

[3] An equitable allocation need not be efficient. Thus, if C were given an end piece and A or B the middle piece in the example, cutpoints could be found such all the players receive, in their own eyes, the same value. However, this value will be less than an equitable allocation if C gets the middle piece and A and B the end pieces.

when the knife reaches the 1/3 point for each. Let the *first* player to call stop be player A. (If two players call stop at the same time, choose one randomly.)

Let A place a mark at the point where it calls stop (the right boundary of piece 1 in the diagram below), and a second mark to the right that bisects the remainder of the cake (the right boundary of piece 2 below). Thereby A indicates the two points that, for it, trisect the cake into pieces 1, 2, and 3:

```
        1               2               3
0---------------|---------------|---------------1.
                A               A
```

Because neither player B nor player C called stop before A did, each of B and C thinks that piece 1 is at most 1/3. They are then asked whether they prefer piece 2 or piece 3. There are three cases to consider:

1. If B and C each prefers a different piece—one player prefers piece 2 and the other piece 3—we are done: A, B, and C can each be assigned a piece that they consider to be at least tied for largest.

2. Assume B and C both prefer piece 2. A referee places a knife at the right boundary of piece 2 and moves it to the left. Meanwhile, A places a knife at the left boundary of piece 2 and moves it to the right in such a way that the amounts of cake traversed on the left and right are equal for A. Thereby pieces 1 and 3 increase equally in A's eyes. At some point, piece 2 will be diminished sufficiently to piece 2'—in either B's or C's eyes—to tie with either piece 1' or piece 3', the enlarged 1 and 3 pieces. Assume B is the first, or the tied-for-first, player to call stop when this happens; then give C piece 2', which it still thinks is the largest or the tied-for-largest piece. Give B the

piece it thinks ties for largest with piece 2' (say, piece 1'), and give A the remaining piece (piece 3'), which it thinks ties for largest with the other enlarged piece (piece 1'). Clearly, each player will think it gets at least a tied-for-largest piece.

3. Assume B and C both prefer piece 3. A referee places a knife at the right boundary of piece 2 and moves it to the right. Meanwhile, A places a knife at the left boundary of piece 2 and moves it to the right in such a way as to maintain the equality, in its view, of pieces 1 and 2 as they increase. At some point, piece 3 will be diminished sufficiently to piece 3'—in either B or C's eyes—to tie with either piece 1' or piece 2', the enlarged 1 and 2 pieces. Assume B is the first, or the tied-for-first, player to call stop when this happens; then give C piece 3', which it still thinks is the largest or the tied-for-largest piece. Give B the piece it thinks ties for largest with piece 3' (say, piece 1'), and give A the remaining piece (piece 2'), which it thinks ties for largest with the other enlarged piece (1'). Clearly, each player will think it got at least a tied-for-largest piece.

Note that which player moves a knife or knives varies, depending on what stage is reached in the procedure. In the beginning, we assume a referee moves a single knife, and the first player to call stop (A) then trisects the cake. But at the next stage of the procedure, in cases (2) and (3), it is the referee and A that move two knives simultaneously, "squeezing" what players B and C consider to be the largest piece until it eventually ties, for one of them, with one of the two other pieces. While Barbanel and Brams show that squeezing can also be used to produce an "almost" envy-free 4-person, 3-cut division (at most one player is envious), absolute envy-freeness eludes them unless up to 5 cuts are allowed, which may, as in Austin's procedure, require combining disconnected pieces.

Earlier, Brams, Taylor, and Zwicker (1997) gave a 4-person, envy-free procedure that requires up to 11 cuts; chore division for 4 players requires even more (16 cuts)

(Peterson and Su, 2002). Because the Brams-Taylor-Zwicker 4-person procedure involves fewer cases than the Barbanel-Brams procedure, it is probably simpler, even though it requires more cuts (11 versus 5).

Beyond 4 players, no procedure is known that yields an envy-free division of a cake unless an unbounded number of cuts is allowed (Brams and Taylor, 1995, 1996; Robertson and Webb, 1998). While this number can be shown to be finite, it cannot be specified in advance—this will depend on the specific cake being divided. The complexity of what Brams and Taylor call the "trimming procedure" makes it of dubious practical value.

To return to the question posed in the title of this section, we now ask how money can help turn the efficiency and envy-freeness of the Barbanel-Brams 3-person, 2-cut procedure into an equitable division as well. The answer is the same as that given for rendering cut-and-choose equitable (section 5), with one major restriction: The money that players put into a pot must be divided equally among them to ensure overall equitability; no other kind of equitability guarantees that envy-freeness will be preserved (Chisholm, 2000).

What each player puts into the pot is the difference between its valuation of its envy-free piece and its valuation of the entire cake (assumed to be the same for all players), divided by n. If, for example, three players value the cake at \$24, and they get envy-free pieces they value at (\$12, \$10, \$8), they must put into the pot (\$4, \$2, \$0), or the differences between their valuations and \$24/3 = \$8.

As in the case of cut-and-choose with money, we assume that the players begin by submitting their value functions to a referee. Then an envy-free cake-cutting procedure, like that of Barbanel and Brams, is implemented, which could as well be done by the referee—knowing the players' value functions—as by the players. Because the players must be truthful in order to ensure envy-freeness, the referee will also know from their

valuations of the pieces they obtain how much each player owes to the pot. Thus, the referee can also implement the money phase of an envy-free procedure.

What distinguishes an *n*-person envy-free procedure from cut-and-choose with money, in which only the chooser contributes to the pot, is that the money from the pot must be divided equally among the *n* players to guarantee envy-freeness (and satisfy the maximin assumption). This is not the case with cut-and-choose, because no matter how much (or little) money is given by the chooser to the cutter under any one of the equitability concepts, both players possess in cake at least 1/2 their valuations. This is a sufficient amount to ensure envy-freeness, whatever equitability concept is used.

This is not the case, however, if there are 3 or more players. Under the Barbanel-Brams procedure, for example, A always thinks that its piece ties with that of B, C, or both. Therefore, it will be envy-free only if the money distributed, above and beyond each player's 1/3 portion, is distributed equally.

This argument holds whatever *n*-person envy-free procedure is implemented. Each player will think, after it has contributed to the pot, that it got exactly 1/*n* of its valuation. After the money from the pot is distributed, each will think its valuation was augmented by the same amount, given that the value of the entire cake is the same for all players.

This means that only overall equitability in the 2-person case (section 5)—whereby the players think they get exactly the same amounts of the entire cake—is applicable in the *n*-person case if envy-freeness is to be preserved. The other kinds of equitability, under which the players may receive different amounts of money, may create envy.

Efficiency may be sacrificed if a procedure uses more than $n-1$ cuts, as we observed in the case of Austin's 2-person procedure (section 2). This is what Brams and Taylor (1996, pp. 150-151) call "C-efficiency" (the C stands for "cut"), or efficiency with respect to $n-1$ parallel, vertical cuts. While the Barbanel-Brams 3-person, 2-cut procedure with money is C-efficient, no 4-person, 3-cut procedure that gives envy-

freeness without money, and therefore equitability and C-efficiency with money, is known.

7. Conclusions

We have described a new 2-person, 1-cut cake-cutting procedure that is efficient, equitable, and envy-free and, therefore, perfect. Neither cut-and-choose nor Austin's procedure yields a division that satisfies all three properties for players that are risk-averse and make maximin choices. Although the new procedure is somewhat more complex than its forbears because it may require the payment of money to one player, each player gets, in the end, at least 1/2 of the cake as it values it and at least 1/2 of the surplus (in cake or money) that remains after the players have received their 50% cake portions.

Moving up one player and one cut, a 3-person, 2-cut division that is envy-free and equitable may not exist, so a procedure for implementing such a division of cake alone is impossible. But with the addition of money, there is a procedure that gives both envy-freeness and equitability as well as efficiency (in the class of 2-cut procedures using parallel, vertical cuts). But whether there exists a 4-person, 3-cut envy-free cake-cutting procedure that, with the addition of money, can be rendered equitable and efficient is an open question.

There are procedures, using more than the minimum number of cuts ($n - 1$), that give envy-free allocations for 4 players, but past 4 players the only procedure known to produce an envy-free division requires an unbounded, though finite, number of cuts. All these procedures can be made equitable with money, but they are not necessarily efficient in the class of procedures that require nonminimal numbers of cuts.

Most disputes over goods, property, or money boil down to two or three parties, so it is pleasing to have procedures that yield efficient, equitable, and envy-free divisions. If there are multiple divisible goods that must be divided, 2-person procedures like

"adjusted winner" (Brams and Taylor, 1996, 1999) seem more applicable than cake-cutting procedures, though Jones shows that adjusted winner can be viewed as a perfect cake-cutting procedure. While the fair division of indivisible goods poses significant new challenges (Brams, Edelman, and Fishburn, 2001), progress has recently been made in finding ways of dividing these fairly (Brams and Fishburn, 2000; Edelman and Fishburn, 2001; Brams, Edelman, and Fishburn, forthcoming; Brams and King, 2003).

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