Abstract

This paper considers the optimality of setting a secret reserve price rather than a public one in an e-ascending auction. We determine the seller’s optimal secret reserve price when the bidders’ values are private and independently distributed. Then, we compare the seller’s interim revenues under the two reserve price policies and we show that the optimal secret reserve price policy can generate higher revenues for the seller only when bidders are risk-averse. This result depends on the bidders’ degree of risk aversion.

1. Introduction

A huge volume of exchange is realized through auctions. In the case of the sale of an indivisible object, four basic types of auctions have been commonly used or analyzed in the literature, the English (or ascending) auction, the Dutch (or descending) auction, the first price sealed bid auction and the second price sealed bid (or Vickrey) auction. Though all these formats can now be found on the auction sites on the Internet\(^1\), the English ascending auction seems to be the format the most commonly designed by the auctioneers. In this type of auction, any bidder can view the current high bid for the item sold and decide to raise it. After submitting his bid, the bidder will see an automatic update of the auction status, showing him whether he is now the current high bidder. He can leave the

\(^1\)See Lucking Riley (2000).
site as the winning bidder and return at any time before the close of the auction
to check on its status again. However, there are different versions of the English
ascending auction. In particular, auctioneers can choose among different policies
concerning the reserve price, i.e. the price below which the item is not sold. There
may be no reserve price or, when there is a reserve price, it may be public
or secret. Thus, on most of English auction sites, bidders are informed when the
reserve price has been met (by changing the color of an icon for example). But a
minority of English-auction sites doesn’t give any information on the level of the
reserve price until the auction ends. What is the interest of keeping the reserve
price secret? Several papers have considered secret reserve prices under different
assumptions in first or second price sealed-bid auctions. Elyakime, Laffont, Loisel
and Vuong (1994) analyze timber’s first-price sealed-bid auctions where the seller
doesn’t reveal her reserve price and show that a public reserve price is better from
the seller’s point of view. When bidders’ signals are affiliated, Vincent (1995)
shows that secret reserve prices in a second-price sealed-bid auction can be better
for the seller: a public reserve price may scare a bidder away because he doesn’t get
to see the signals of others who are screened out by the reserve price. Ashenfelter
(1989) considers the interest of a secret reserve price in the ascending auctions
used by auction-houses like Christie’s or Sotheby’s. He argues that setting a secret
reserve price in English auctions for wine and arts may thwart collusion between
bidders but concludes that the results on the optimal English auction do not give
arguments in favor of a secret reserve price because bidding up until one’s private
value is a dominant strategy irrespective of the reserve price policy. The issue has
also been considered empirically. In an empirical study of E-Bay coin auctions,
Bajari and Hortacsu (2000) observe that the sellers of items with high book values
tend to use a secret reserve price while the sellers of items with low book values
used a posted reserve price. In a field experiment in E-Bay auctions, Kathar and
Lucking-Reiley (2000) find that secret reserve prices make sellers worse off by
reducing the probability of an exchange, deterring serious bidders from entering
the auction and lowering the expected price of the auction.

This paper addresses the following issue: can an optimal secret reserve price
be better than an optimal public one for the seller in an e-ascending auction? The
introduction of a reserve price allows the seller to capture some of the informational
rent of the winning bidder. When the seller and the buyers are risk neutral and
when the values are private and independent, the optimal auction implies a public
reserve price, greater than the seller’s value for the object. Ascending auctions
are often referred in the literature as to open second price auctions, which are
optimal auctions. In the Japanese ascending auction where prices are increased continuously, with independent private values, it is a dominant strategy to bid up until the price reaches one’s evaluation. The next-to-last person will drop out when the price reaches his value and the bidder with the highest value will pay the price attained when he becomes the sole bidder, i.e. the second highest value. In a second price auction, truth telling is a weakly dominant strategy, then the highest bidder wins and pays the second highest value. This is why these auctions are considered as equivalent. In the e-ascending auction we consider, the bidder pays his bid. When the reserve price is public, this does not change the equivalence with a second price auction because each bidder can always decide to increase the standing high bid by \( \varepsilon \). However, when the reserve price is secret, even with private independent values, we cannot consider anymore that the ascending auction is equivalent to the second price auction. In the first mechanism, bidders have to take the uncertainty on the reserve price into account when they decide to increase the current high bid. This can induce a bidder to increase the current high bid by more than \( \varepsilon \). In a second price auction, the fact that truth telling is a dominant strategy is independent of the seller’s reserve price policy.

In the following, we first analyze the seller’s and buyers’ optimal strategies in the ascending auction with a secret reserve price when the values are uniformly distributed. Then we compare both reserve price policies from the two points of view of the expected revenue and of the probability of exchange. We find that the optimal secret reserve price policy is dominated when the bidders are risk neutral but can generate higher revenues when bidders are risk-averse because the optimal response of the bidders to the secret reserve price is higher. The last result depends on the bidders’ degree of risk aversion. However we show that, for the same number of bidders, using a secret reserve price always decreases the risk of non-exchange.

2. Seller’s and buyers’ optimal strategies with a secret reserve price

We consider that a risk-neutral seller having an indivisible object to sell designs an ascending auction without incremental price and commits not to re-auction the object if the sale fails\(^2\). She faces \( n \) potential risk neutral buyers. We assume

\(^2\)This allows not to consider the influence of resale on the strategy to set a public or secret reserve price.
that bidders’ reservation values \(v_i\) are private information and are independently drawn from the same cumulative function, \(F(.)\), with a positive and continuously differentiable density function \(f(.)\) on \([0,\infty]\). We also assume that the seller has a private reservation value \(v_0\), drawn from a cumulative function \(H(.)\) on \([0,\infty]\). \(F(.)\) and \(H(.)\) are common knowledge. To ensure commitment on the reserve price, we assume that the seller put it secretly in an envelope which is opened only after the auction stops. Moreover, we assume that there is a closing time for the auction but that the highest bid is always submitted before the closing time.

In this ascending auction with a public reserve price, the weakly dominant strategy of each bidder consists in rising the current high bid (by \(\varepsilon\)) as long as it is lower than one’s private value. Then the auction stops at a price equal to the maximum of the second highest private value and the public reserve price if the latest is lower than the highest private value. If the reserve price is higher than the highest private value, the object is not awarded. When the reserve price is secret, the optimal strategy is less straightforward because each bidder has to anticipate the secret reserve price. As a matter of fact, a secret reserve price auction induces a Bayesian game between the seller and the bidders.

2.1. The seller’s optimal strategy

The seller chooses the best reserve price function \(r_s(v_0)\) in response to the bidders’ optimal bidding strategies, whereas each bidder chooses his optimal strategy in response to the seller’s and the other bidders’ optimal strategies. At round \(t\), assume that \(b\) is the current high bid. Bidder \(j\) has an incentive to rise \(b\) if \(v_j > b\) and he may bid \(b + \varepsilon\) or higher than \(b + \varepsilon\), taking the uncertainty on the seller’s reserve price into account. At this round, his optimal bidding strategy consists in choosing \(b^*(v_j)\) solution of

\[
\text{Max} \ (v_j - b(v_j)) \cdot \Pr(b(v_j) > r_s(v_o))
\]

subject to \(b(v_j) > b\)

If \(b\) is lower than \(b(v_j) \in \text{Arg max} \ (v_j - b(v_j)) \cdot \Pr(b(v_j) > r_s(v_o))\), \(b^*(v_j) = b(v_j)\). If \(b > b(v_j)\), then \(b^*(v_j) = b + \varepsilon\). Given this optimal strategy, the timing of the auction is the following: the first arrival \(i\) will submit a bid \(b(v_i)\) which is the optimal response to the secret reserve price\(^3\). If this current high bid is lower than \(b(v_j)\), bidder \(j\) will post \(b(v_j)\) as standing high bid. Then the auction price will

\(^3\)In the following, we assume that \(b(v_i)\) is a symmetric, monotone and increasing function of \(v_i\).
increase until \(b(v(1))\) where \(v(1)\) is the highest private value. If \(v(2)\), the second highest private value is lower than \(b(v(1))\), the auction stops at \(b(v(1))\) and the winner pays this price if \(b(v(1)) \geq r_s(v_0)\). If \(b(v(1)) < r_s(v_0)\), the item is not sold. On the other hand, if \(v(2) > b(v(1))\), the auction will continue as an usual English ascending auction until \(v(2) + \epsilon\) and the bidder with \(v(1)\) wins and pays \(v(2) + \epsilon\) if \(v(2) \geq r_s(v_0)\). If \(v(2) < r_s(v_0)\), the item is not sold as in the previous case.

We can now consider the seller’s optimal strategy. Taking the bidders’ strategies as given, when the seller deviates by \(r_s(w_0)\) from her strategy \(r_s(v_0)\), her expected revenue \(ER_s(v_0, w_0)\) is, when \(b(\overline{v}) > r_s(w_0)\):

\[
ER_s(v_0, w_0) = v_0[nF(r_s(w_0))^{n-1}F(b^{-1}(r_s(w_0))) - (n-1)F(r_s(w_0))^n]
\]

\[+
\int_{v(1) : b(v(1)) \geq r_s(w_0)}^{\overline{v}} \int_{v(2):0}^{b(v(1))} b(v(1))n(n-1)F(v(2))^{n-2}f(v(2))dv(2)f(v(1))dv(1)
\]

\[+
\int_{v(1) : r_s(w_0)}^{b^{-1}(r_s(w_0))} \int_{v(2):r_s(w_0)}^{v(1)} v(2)n(n-1)F(v(2))^{n-2}f(v(2))dv(2)f(v(1))dv(1)
\]

\[+
\int_{v(1) : b(v(1)) \geq r_s(w_0)}^{\overline{v}} \int_{v(2):b(v(1))}^{v(1)} v(2)n(n-1)F(v(2))^{n-2}f(v(2))dv(2)f(v(1))dv(1)
\] (2.1)

The first term corresponds to the expected revenue when the good is not sold, i.e. when \(v(1) < r_s(w_0)\) or \(v(1) \geq r_s(w_0)\) and \(v(2) < r_s(w_0)\) and \(b(v(1)) < r_s(w_0)\). The second term corresponds to the expected revenue when the price paid is \(b(v(1))\), which occurs when \(v(2) < b(v(1))\) and \(b(v(1)) > r_s(w_0)\). The third and the fourth terms correspond to the expected revenue when the price paid is \(r_s(w_0)\), which occurs when \(b(v(1)) \leq r_s(w_0) \leq v(2)\) (third term) or when \(r_s(w_0) \leq b(v(1)) \leq v(2)\) (fourth term).

When \(b(\overline{v}) \leq r_s(w_0)\), i.e. \(\overline{v} \leq b^{-1}(r_s(w_0))\), the optimal response \(b(v_i)\) can never win against the secret reserve price and the ascending auction with a secret reserve price gives always a price equal to the second highest value when the item is sold. The expected revenue is equal

\[
ER_s(v_0, w_0) = v_0[nF(r_s(w_0))^{n-1} - (n-1)F(r_s(w_0))^n]
\]

\[+
\int_{v(1):r_s(w_0)}^{v(1)} \int_{v(2):r_s(w_0)}^{v(1)} v(2)n(n-1)F(v(2))^{n-2}f(v(2))dv(2)f(v(1))dv(1)
\] (2.2)
The strategy \( r_s(v_0) \) is optimal if no deviation is profitable. Using the first order condition, we obtain the following proposition:

**Proposition 2.1.** Under a secret reserve price, it is a dominant strategy for the seller to set a reserve price equal to her true valuation \( v_0 \).

\[ r_s(v_0) = v_0 \] (2.3)

**Proof.** See appendix A.

From proposition 2.1, the seller has no incentive to change her mind when she discovers the buyers’ bids. Her strategy is independent of the buyers’ strategies. The result is the same as in Elyakime, Laffont, Loisel and Vuong (1994), who analyze a first price auction. The seller has no interest to lie to herself when she determines the secret reserve price.

### 2.2. The bidders’ optimal strategy

We can now analyze the bidders’ symmetric optimal strategy. As described previously, the strategy has two components. Firstly, taking the uncertainty on the secret reserve price into account, bidder \( i \) has an incentive to raise the current high bid until \( b(v_i) \). Then, the bidder with the highest value bids \( b(v_{(1)}) \) and each bidder \( j \) commits to the usual ascending auction strategy if \( b(v_{(1)}) < v_j \).

Formally, \( b(v_i) \) maximizes:

\[ \Pi(v_i, w) = (v_i - b(w)). \Pr(b(w) \geq r_s(v_0) = v_0) = (v_i - b(w)).H(b(w)) \]

Then \( b(v_i) \) is the symmetric strategy solution of \( \frac{\partial \Pi(v_i, w)}{\partial w} \big|_{w=v_i} = 0 \). We obtain an implicit solution defined by:

\[ b(v_i) = v_i - \frac{H(b(v_i))}{h(b(v_i))} \] (2.4)

which gives the optimal response of bidder \( i \) to the uncertainty on the reserve price.
3. Secret or public reserve price with risk neutral bidders

The interest of a reserve price policy for the seller is to capture some of the informational rent of the winning bidder. The optimal reserve price is the result of a trade-off between the seller’s expected revenue and the ex post efficiency, and then the probability of exchange. To consider these two issues, we assume that each bidder’s private information $v_i$ and the seller’s private value $v_0$ are respectively drawn independently from the uniform $[0,1]$ distributions.

3.1. Seller’s expected revenue comparison

Under our assumptions, Riley and Samuelson (1981) have shown that the public reserve price is $r_p(v_0) = \frac{v_0 + 1}{2}$ and the seller’s expected revenue at the interim stage is:

$$ ER_p(v_0) = \frac{(v_0 + 1)^{n+1}}{2^n(n+1)} + \frac{n - 1}{n+1} $$ (3.1)

When the reserve price is secret, we obtain from (2.4)

$$ b(v_i) = \frac{v_i}{2} $$

Replacing $b(v_i)$ in (2.1) and (2.2)$^4$, from (2.3), we have

$$ ER_s(v_0) = v_0^{n+1} + \frac{1}{2^n(n+1)} + \frac{n - 1}{n+1} \text{ for } v_0 \in [0,\frac{1}{2}] $$ (3.2)

$$ ER_s(v_0) = v_0^n - \frac{n - 1}{n + 1} v_0^{n+1} + \frac{n - 1}{n + 1} \text{ for } v_0 \in [\frac{1}{2}, 1] $$ (3.3)

If we denote $\Delta R_N(v_0) = ER_p(v_0) - ER_s(v_0)$, from (3.1), (3.2) and (3.3), we have

$$ \Delta R_N(v_0) = \left(\frac{(v_0 + 1)^{n+1}}{2^n(n+1)} - v_0^{n+1} - \frac{1}{2^n(n+1)} \right) \text{ for } v_0 \in [0,\frac{1}{2}] $$ (3.4)

$$ \Delta R_N(v_0) = \left(\frac{(v_0 + 1)^{n+1}}{2^n(n+1)} + \frac{n - 1}{n + 1} v_0^{n+1} - v_0^n \right) \text{ for } v_0 \in [\frac{1}{2}, 1] $$ (3.5)

and we obtain proposition 3.1.

$^4$When $v_0 < \frac{1}{2}$, (2.1) applies whereas (2.2) applies when $v_0 > \frac{1}{2}$. 

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Proposition 3.1. When the values are uniformly distributed, the expected interim revenue in an ascending auction is strictly higher when the reserve price is public than when it is secret for $v_0 \in (0,1)$. When $v_0 = 0$, the expected interim revenues are equal. When $v_0 = 1$, the secret and the public reserve price are equal and higher than the bidders valuations and the expected interim revenue are equal to 1 (the item is never sold).

Proof. See appendix 2. ■

In the ascending auction with a public reserve price, when a bidder has a value for the object higher than the seller’s value, the reserve price is binding only when the bidder with the second highest value drops out before the reserve price. The optimal reserve price is the result of the following trade off. On one hand, the object may not be sold even if this highest value is higher than the seller’s value. This is the case when $v_0 < v(1) < r_p(v_0)$. On the other hand, when the highest value is higher than the seller’s reserve price, the price paid is $\max\{r_p(v_0), v(2)\}$. It is at least equal to the reserve price and is higher than the price paid without reserve price or with a reserve price equal to $v_0$. With a secret reserve price, the object may not be sold even if the highest value is higher than the seller’s reserve price, i.e. even if $r_s(v_0) = v_0 < v(1)$. This is the case when $r_s(v_0) < v(1)$ and $v(2)$ and $b(v(1)) < r_s(v_0)$ when $v_0 \in [0, \frac{1}{2}]$ and when $v(2) < r_s(v_0) < v(1)$ when $v_0 \in [\frac{1}{2}, 1]$. When the object is sold, i.e. when $\max\{b(v_1), v(2)\} > r_s(v_0) = v_0$, the price paid is $\max\{b(v_1), v(2)\}$, where $b(v_1)$ is the response of the bidder with the highest value to the seller’s secret reserve price when $v_0 \in [0, \frac{1}{2}]$ and the price paid is $v(2)$ when $v_0 \in [\frac{1}{2}, 1]$. The probability of not selling the object and the price paid are different and the optimal public reserve price policy dominates the optimal secret reserve price policy.

Under a private values assumption, the ascending auction with a public reserve price is equivalent to a second-price sealed-bid auction where “truth telling” is a weakly dominant strategy equilibrium. As keeping the reserve price secret does not change the optimal strategy of the bidders in a second price auction, from proposition (3.1), we can state the following corollary:

Corollary 3.2. When values are private and uniformly distributed and when the reserve price is secret, the second price auction results in an expected revenue greater than the expected revenue obtained with a secret reserve price ascending auction.
In practice, mechanism designers often use a proxy bidding system. The method consists in asking the buyers to give the highest amount they are willing to pay. Their bid will be kept secret and will not be used unless needed. The system will automatically increase a bid by the next increment until the maximum bid of a buyer is reached. eBay explains the proxy bidding system as follows: “Everyone has a little magical elf” aka proxy” to bid for them... all you need to do is tell your elf the most you want to spend for the item and he’ll sit there and outbid other elves for you, until his limit is reached”. Then, in a private-values model of auction bidding, this mechanism can be considered as a second-price sealed-bid auction. Therefore, from corollary (3.2), we can conclude that the proxy bidding system is better for the auctioneer than the ascending auction when the reserve price is secret and when bidders are risk neutral.

3.2. Probability of exchange

As we have seen in the previous section, when \( v_0 \in [0, \frac{1}{2}] \), the probability to award the item is not the same when the price is secret or when it is public. If the public reserve price policy is better when the seller seeks to maximize her expected revenue, the following proposition suggests that it may be worse from the awarding point of view. In a e-ascending auction with a public reserve price, the probability that the item is not awarded is

\[
F(\ r_p(v_0)) = \left( \frac{v_0 + 1}{2} \right)^n
\]

whereas with a secret reserve price, this probability is

\[
(nF(r_s(v_0))^{n-1}F(b^{-1}(r_s(v_0))) - (n-1) F(r_s(v_0))^n = v_0^n(n + 1)
\]

**Proposition 3.3.** In the e-ascending auction, when \( v_0 \in [0, \frac{1}{2}] \), the probability that the item is not awarded is greater with a public reserve price than with a secret reserve price for \( v_0 < \frac{1}{2(n+1)} \). For a given number of bidders.

**Proof.** This result comes directly from the comparison of \( \left( \frac{v_0 + 1}{2} \right)^n \) and \( v_0^n(n + 1) \).

We can note that \( \frac{1}{2(n+1)^{\frac{n}{2} - 1}} \) is increasing in \( n \). Then, the greater \( n \), the better the secret reserve price policy from the probability of exchange point of view. When \( n = 2 \), proposition 2.4 is verified for \( v_0 < 0.4058 \) and when \( n = 3 \), it is verified for \( v_0 < 0.4598 \). As soon as \( n = 4 \), \( v_0 \leq \frac{1}{2} < \frac{1}{2(n+1)^{\frac{n}{2} - 1}} \) and the probability to award the object is always greater with a secret reserve price.
4. Secret or public reserve price with risk averse bidders

The introduction of risk aversion for the bidders may inverse the result obtained in the case of neutrality, because risk aversion affects the bidders’ response to the secret reserve price. We have shown previously that the seller’s reserve price policy is independent of the assumption about bidders risk aversion. Then, it is the same as in the case of risk-neutral bidders, i.e. \( r_s(v_0) = v_0 \). When the reserve price is public, bidders stay (or bid) in the auction up to the point where the price reaches their true valuation. But, when the reserve price is secret, the bidder’s strategy \( b(v_i) \) (called here \( b_a(v_i) \)) is more aggressive than in the case of risk-neutral bidders.

Assume that the bidder’s preferences can be described by the utility function \( u(v) = \frac{v^{1-\theta}}{1-\theta} \) with \( 0 < \theta < 1 \), where \( \theta \) measures the degree of constant relative risk aversion.

For simplicity, assume as previously that each bidder’s private information \( v_i \) and the seller’s private value \( v_0 \) are drawn independently from the uniform \([0, 1]\) distribution. As in the neutrality case, the optimal response to the secret reserve price, \( b_a(v_i) \), is the symmetric strategy solution of 

\[
\frac{\partial \Pi(v_i, w)}{\partial w} |_{w=v_i} = 0
\]

Then, we obtain the following lemma:

**Lemma 4.1.** When the reserve price is secret, the risk-averse bidder’s optimal strategy is \( b_a(v_i) = \frac{1}{2-\theta} v_i \), and is more aggressive than when the bidders are risk neutral.

Proof.:

\[
\frac{\partial \Pi(v_i, w)}{\partial w} |_{w=v_i} = \frac{2-\theta}{1-\theta} b'_a(v_i) \left[ (v_i - b_a(v_i))^{1-\theta} - (1-\theta)(v_i - b_a(v_i))^{-\theta} b_a(v_i) \right] = 0
\]

and we obtain

\[
b_a(v_i) = \frac{v_i}{2-\theta}
\]  

(4.1)

We can now compute the seller’s expected revenue with a secret reserve price when bidders are risk averse. From (2.1), (2.2) and (2.3) and replacing \( b(v_i) \) defined in (4.1), we obtain:
i) If \( v_o \in [0, \frac{1}{2}] \)

\[
ER_s^A(v_o) = v_o^{n+1}(1 - \frac{n\theta}{n+1}) + \frac{1}{n+1} \frac{1}{(2-\theta)^n} + \frac{n-1}{n+1} \forall \theta \in (0, 1)
\]

\[\text{ii) If } v_o \in [\frac{1}{2}, 1] \]

\[
ER_s^A(v_o) = v_o^{n+1}(1 - \frac{n\theta}{n+1}) + \frac{1}{n+1} \frac{1}{(2-\theta)^n} + \frac{n-1}{n+1} \forall \theta \in [\overline{\theta}, 1)
\]

with \( \overline{\theta} = 2 - \frac{1}{v_0} \).

We can again compare the two reserve price policies by studying the sign of \( \Delta R_A(v_o) = ER_p(v_o) - ER_s^A(v_o) \). As bidders’ strategies in the ascending auction are independent of their attitude toward risk when the reserve price is public, the seller’s expected revenue is given by (3.1). Then, we have:

i) If \( v_o \in [0, \frac{1}{2}] \)

\[
\Delta R_A(v_o) = \frac{(v_0 + 1)^{n+1}}{2^n(n+1)} - v_o^{n+1}(1 - \frac{n\theta}{n+1}) - \frac{1}{n+1} \frac{1}{(2-\theta)^n} \forall \theta \in (0, 1) \quad (4.2)
\]

\[\text{ii) If } v_o \in [\frac{1}{2}, 1] \]

\[
\Delta R_A(v_o) = \frac{(v_0 + 1)^{n+1}}{2^n(n+1)} - v_o^{n+1}(1 - \frac{n\theta}{n+1}) - \frac{1}{n+1} \frac{1}{(2-\theta)^n} \text{ if } \theta \in [\overline{\theta}, 1)
\]

\[
\Delta R_A(v_0) = \frac{(v_0 + 1)^{n+1}}{2^n(n+1)} + \frac{n-1}{n+1} v_o^{n+1} - v_o^n \text{ if } \theta \in (0, \overline{\theta}] \quad (4.4)
\]

A first result can easily be obtained when the bidders are highly risk averse.

**Proposition 4.2.** When \( \theta \) tends to 1 (bidders are highly risk averse), it is a weakly dominant strategy for the seller to choose a secret reserve price auction, irrespective of her value \( v_0 \).

\[\text{5In this case, (2.1) applies}
\]

\[\text{6When } \theta < \overline{\theta}, b(\pi) < v_0 \text{ and (2.2) applies whereas when } \theta > \overline{\theta}, b(\pi) > v_0 \text{ and (2.1) applies.}\]
Proof. : When \( \theta \to 1 \), we have \( b_{a}(v_{i}) \to v_{i} \). Then the seller can extract all the informational rent from the bidders. Moreover, \( r_{s}(v_{0}) = v_{0} \). Then there is no risk of ex post inefficiency. Therefore, the ascending auction with a secret reserve price will always be better than with a public reserve price which implies a risk of ex post inefficiency.

Moreover, we can easily verify that when \( v_{0} = 0 \), (4.2) is always strictly negative with \( \theta \in (0, 1) \). Then when the good has no value for the seller, it is always optimal to set a secret reserve price (equal to 0). When \( v_{0} > 0 \), we can consider the variation of \( \Delta R(v_{o}) \) when \( \theta \) varies from 0 to 1 and we obtain the following proposition:

Proposition 4.3. When \( v_{0} > 0 \), there is a threshold \( \hat{\theta}(n, v_{0}) \geq \bar{\theta} \) such that when the degree of constant relative risk aversion is lower than \( \hat{\theta}(n, v_{0}) \), the public reserve price policy dominates the secret reserve price policy, whereas when the degree of constant relative risk aversion is higher than \( \hat{\theta}(n, v_{0}) \), it is the secret reserve price policy which dominates.

Proof. Differentiating (4.3), we obtain

\[
\frac{\partial \Delta R(v_{o})}{\partial \theta} = \frac{n}{n+1} \left( v_{0}^{n+1} - \frac{1}{(2-\theta)^{n+1}} \right) \tag{4.5}
\]

For a given \( v_{o} \), two cases must be considered.

i) \( v_{o} \in [0, \frac{1}{2}] \). As the derivative is negative for \( v_{o} < \frac{1}{(2-\theta)} \) and as \( \frac{1}{2} \leq \frac{1}{(2-\theta)} \) \( \forall \theta \in [0, 1) \), \( \Delta R(v_{o}) \) is decreasing in \( \theta \). As \( \Delta R(v_{o}) \) is positive for \( \theta = 0 \) and negative for \( \theta \to 1 \), there is a threshold \( \hat{\theta}(n, v_{o}) \) such that \( \Delta R(v_{o}) > 0 \) for \( \theta < \hat{\theta}(n, v_{0}) \) and \( \Delta R(v_{o}) < 0 \) for \( \theta > \hat{\theta}(n, v_{0}) \).

ii) If \( v_{o} \in [\frac{1}{2}, 1] \), \( \Delta R(v_{o}) \) is defined by (4.3) for \( \theta \in [\bar{\theta}, 1) \). For \( \theta = \bar{\theta} \), \( \frac{\partial \Delta R(v_{o})}{\partial \theta} = 0 \) and for \( \theta > \bar{\theta}(v_{o}) \), \( \frac{\partial \Delta R(v_{o})}{\partial \theta} < 0 \). Then \( \Delta R(v_{o}) \) is decreasing in \( \theta \). As \( \Delta R(v_{o}) \) is positive for \( \theta = \bar{\theta} \) (see appendix C) and negative for \( \theta \to 1 \), as in i), there is a threshold \( \hat{\theta}(n, v_{o}) \) such that \( \Delta R(v_{o}) > 0 \) for \( \theta < \hat{\theta}(n, v_{0}) \) and \( \Delta R(v_{o}) < 0 \) for \( \theta > \hat{\theta}(n, v_{0}) \), with \( \hat{\theta}(n, v_{o}) > \bar{\theta}(v_{o}) \). For \( \theta \in (0, \bar{\theta}), \Delta R(v_{o}) \) is defined by (4.4) and is positive.

The following table gives \( \bar{\theta} \) and \( \hat{\theta}(n, v_{0}) \) for different values of \( v_{o} \) and \( n \).
From the table, we see that the threshold $\bar{\theta}(n, v_0)$ increases with seller's value $v_0$ for a given $n$ and decreases with the number of bidders for a given $v_0$. Then the lower the seller's value and the greater the competition, the more interesting the secret reserve price policy when the bidders are risk averse.

5. Conclusion

This paper gives us a first theoretical explanation of the use of a secret reserve price in an e-ascending auction. In contrast with the preconceived idea that ascending auction can always be modelized as a second price sealed bid auction in an independent private value environment, our paper shows that this is not always the case. In fact, when the reserve price is secret, we show that second price auctions and ascending auctions are not equivalent. When bidders are risk-neutral, the second price auctions with a public or a secret reserve price always result in an expected revenue greater than an ascending auction with a secret reserve price. However, when bidders are risk-averse, an optimal reserve price’s ascending auction can generate higher revenues for the seller. In particular, the lower the seller’s reservation value, the more profitable the secret reserve price, even when the bidders’ degree of constant relative risk aversion is low. In the future, we wish to confirm these results in more general contexts in a field-experiment.

A. Proof of proposition 2.1

When $b(v) > r_s(w_0)$, the seller’s expected revenue can be written

$$ER_s(v_0,w_0) = v_0 \left[ nF(r_s(w_0))^{n-1}F(b^{-1}(r_s(w_0))) - (n-1)F(r_s(w_0))^n \right]$$

$$+ \int_{v_1:v_1 \geq r_s(w_0)} \int_{v_2:0} b(v_1) n(n-1)F(v_2)^{n-2}f(v_2)dv_2f(v_1)dv_1$$
The first order condition is equal to
\[ r'_s(v_0) \left[ v_0 \left[ n(n - 1)F(r_s(v_0))^{n-1}F(b^{-1}(r_s(v_0)))f(r_s(v_0)) \right. \right. \\
- b^{-1}r_s(v_0)nF(r_s(v_0))^{n-1}f(b^{-1}(r_s(v_0))) - (n - 1)nF(r_s(v_0))^{n-1}f(r_s(v_0)) \bigg] \\
- nF'(r_s(v_0))^{n-1}r_s(v_0)f(r_s(v_0)) - nF(r_s(v_0))^{n-1}F(b^{-1}(r_s(v_0))) \\
+ nF(r_s(v_0))^{n-1} - n(n - 1)r_s(v_0)F(r_s(v_0))^{n-2}F(b^{-1}(r_s(v_0)))f(r_s(v_0)) \\
- b^{-1}r_s(v_0)nr_s(v_0)F(r_s(v_0))^{n-1}f(b^{-1}(r_s(v_0))) + n^2F(r_s(v_0))^{n-1}r_s(v_0)f(r_s(v_0)) \\
+ nF(r_s(v_0))^{n-1}F(b^{-1}(r_s(v_0))) - nF(r_s(v_0))^{n-1} \bigg] \\
\]

\[ = \ nF(r_s(v_0))^{n-1}r'_s(v_0)(v_0 - r_s(v_0)) \\
\left[ - b^{-1}r_s(v_0)f(b^{-1}(r_s(v_0))) - (n - 1)f(r_s(v_0))(1 - F(b^{-1}(r_s(v_0)))^{n-1}) \right] \]

for \( w_o = v_o \)

Solving

\[ \frac{\partial ER_s(v_0, w_0)}{\partial w_0} \bigg|_{w_0 = v_0} = 0 \]

we obtain

\[ r_s(v_0) = v_0 \]

When \( b(\bar{\pi}) \leq r_s(w_0) \), the seller’s expected revenue is equal to

\[ v_0 \left[ nF(r_s(w_0))^{n-1} - (n - 1)F(r_s(w_0))^n \right] + \bar{\pi} + (n - 1)r_s(w_0)F(r_s(w_0))^n \\
+ (n - 1) \int_{r_s(w_0)}^{\bar{\pi}} F(v)^n \, dv - nr_s(w_0)F(r_s(w_0))^{n-1} - n \int_{r_s(w_0)}^{\bar{\pi}} F(v)^{n-1} \, dv \]

The first order condition \( \frac{\partial ER_s(v_0, w_0)}{\partial w_0} \bigg|_{w_0 = v_0} \) is equal to

\[ (v_0 - r_s(v_0))(n(n - 1)f(r_s(v_0)) - n(n - 1)F(r_s(v_0))f(r_s(v_0))) \]

and is equal to 0 for \( v_0 = r_s(v_0) \).

When \( b(\bar{\pi}) \leq r_s(w_0) \), the seller’s expected revenue is equal to

\[ v_0 \left[ nF(r_s(w_0))^{n-1} - (n - 1)F(r_s(w_0))^n \right] + \bar{\pi} + (n - 1)r_s(w_0)F(r_s(w_0))^n \\
+ (n - 1) \int_{r_s(w_0)}^{\bar{\pi}} F(v)^n \, dv - nr_s(w_0)F(r_s(w_0))^{n-1} - n \int_{r_s(w_0)}^{\bar{\pi}} F(v)^{n-1} \, dv \]

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The first order condition $\frac{\partial E_R(s,w,u)}{\partial w_0} |_{w_0 = v_0}$ is equal to

$$(v_0 - r_s(v_0))(n(n-1)f(r_s(v_0)) - n(n-1)F(r_s(v_0))f(r_s(v_0)))$$

and is equal to 0 for $v_0 = r_s(v_0)$.

**B. Proof of proposition 3.1**

For $v_0 \in [0, \frac{1}{2}]$, to demonstrate proposition 3.1, we study the sign of the first order derivative of $\Delta R(v_0)$ with respect to $v_0$

$$\frac{\partial \Delta R_N(v_0)}{\partial v_0} = \Delta R_N(v_0)' = (v_0 + 1)^n - 2^n(n+1)v_0^n$$

For $n = 2$, $\Delta R_N(v_0)' > 0$ for $v_0 < 0.4$, $\Delta R_N(v_0)' = 0$ for $v_0 = 0.4$ and $\Delta R_N(v_0)' < 0$ for $v_0 > 0.4$. As $\Delta R_N(v_0) = 0$ and $\Delta R_N(\frac{1}{2}) > 0$, $\Delta R_N(v_0) > 0 \forall v_0 \in (0, \frac{1}{2}]$. For $n = 3$, $\Delta R_N(v_0)'$ is positive if $v_0 < 0.46$ and negative if $v_0 > 0.46$. As $\Delta R_N(v_0) = 0$ and $\Delta R_N(1) > 0$, $\Delta R_N(v_0) > 0 \forall v_0 \in (0, \frac{1}{2})$. For $n = 4$, $\Delta R_N(v_0)' > 0 \forall v_0 \in [0, \frac{1}{2}]$ and we show that as soon as $\Delta R_N(v_0)' > 0 \forall v_0 \in [0, \frac{1}{2}]$ for a given $n - 1$, it is positive for $n$. Then as $\Delta R_N(v_0) = 0$ and as the first derivative is positive, $\Delta R_N(v_0) > 0 \forall v_0 \in (0, \frac{1}{2}] \forall n$. Assume $(v_0 + 1)^n - 2^n n v_0^{n-1} > 0$. Multiplying by $(v_0 + 1)^n$, we have $(v_0 + 1)^{2n} > 2^n n v_0^n + 2^n n v_0^{n-1} = A$. To show that $(v_0 + 1)^n > 2^n(n + 1)v_0^n = B$, it is sufficient to show that $A > B$, i.e. $2^n v_0^{n-1}(n - (n + 2)v_0) > 0$. This is true if $v_0 < \frac{n}{n+2}$, which is verified for any $n \geq 3$.

For $v_0 \in [\frac{1}{2}, 1]$, $\Delta R_N(v_0) = \frac{(v_0 + 1)^{n+1} + 2^n(n-1)v_0^n - 2^n(n+1)v_0^n}{2^n(n+1)}$. It can be shown by recurrence that the numerator is always positive for $v_0 \in [\frac{1}{2}, 1]$. It is positive for $n = 2$ and $n = 3$. Let us assume that it is positive for $n - 1$. This implies that $(v_0 + 1)^n > 2^n v_0^{n-1}(n - (n - 2)v_0)$. Multiplying the previous inequality by $(v_0 + 1)^n$, we obtain $(v_0 + 1)^{2n}v_0^{n-1}(n - (n - 2)v_0) = A$ and we want to show that $(v_0 + 1)^{n+1} > 2^n v_0^n((n+1)-(n-1)v_0) = B$. As $A - B = 2^n v_0^{n-1} n(1-v)^2 > 0 \forall v_0$, it is verified.

**C. Element of proof of proposition 4.3**

$\Delta R_A(v_0) \rightarrow \frac{(v_0 + 1)^{n+1}}{2^n(n+1)} - \frac{v_0^{n+1}}{n+1} - \frac{1}{n+1} = \frac{(v_0 + 1)^{n+1} - 2^n(v_0^{n+1} + 1)}{2^n(n+1)}$ when $\theta \rightarrow 1$. As $(v_0 + 1)^{n+1} - 2^n(v_0^{n+1} + 1)$ is strictly increasing in $v_0$ and equal to 0 when $v_0 = 1$, 16
\[ \Delta R_A(v_0) < 0 \text{ when } \theta \to 1 \text{ for } v_0 < 1. \]

To show that \((v_0 + 1)^{n+1} - 2^n(v_0^{n+1} + 1)\) is strictly increasing in \(v_0\), we consider the first order derivative of \((v_0 + 1)^{n+1} - 2^n(v_0^{n+1} + 1)\) in \(v_0\). It is equal to \((n+1)((v_0 + 1)^n - 2^n v_0^n) = (n+1)(\sum_{j=0}^{n} (\frac{n}{j})(v_0^j - v_0^n))\).

As \(v_0 \leq 1\), \(v_0^j > v_0^n\) for \(j < n\) and the first order derivative is positive.

If \(v_0 \in [0, \frac{1}{2}]\), \(\Delta R_A(v_0) \to \frac{(v_0+1)^{n+1}}{2^{n(n+1)}} - v_0^{n+1} - \frac{1}{n+1} \frac{1}{2^n}\) when \(\theta \to 0\), i.e \(\Delta R(v_0)\) tends to the expression obtained in the case of risk neutral bidders, which is positive.

If \(v_0 \in [\frac{1}{2}, 1]\), \(\Delta R_A(v_0) = \frac{(v_0+1)^{n+1}}{2^{n(n+1)}} + \frac{(n-1)v_0^n}{n+1} - v_0^n = \frac{(v_0+1)^{n+1}+2^n(n-1)v_0^n-2^n(n+1)v_0^n}{2^n(n+1)}\) when \(\theta = 2 - \frac{1}{v_0}\) and is equal to \(\Delta R_N(v_0)\) when \(v_0 \in [\frac{1}{2}, 1]\), which is positive (see appendix B).

References


