Differentiated duopoly with asymmetric costs: new results from a seminal model

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Abstract

We compare Bertrand and Cournot equilibria in a differentiated duopoly with linear demand and cost functions. Our basic framework is the Singh and Vives (1984) model, focusing on substitute goods, but allowing for a wider range of cost asymmetry between firms. Our main finding is that, with high degrees of cost asymmetry and/or low degrees of product differentiation, the efficient firm’s and the industry profits are higher under Bertrand than under Cournot competition. This contrasts with the Singh and Vives’s result that with substitute goods both firms earn higher profits under Cournot than under Bertrand competition. We also show that, contrary to the standard result with symmetric costs, the efficient firm may have a local incentive to reduce the degree of product differentiation under both forms of competition.

Key Words: Cost asymmetry; product differentiation; duopoly; Bertrand; Cournot.

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1 Introduction

In a seminal paper, Singh and Vives (1984) show that, in a differentiated duopoly, quantity competition entails higher prices and profits than price competition, whereas quantities and social welfare are higher under price competition.\textsuperscript{1} To obtain these results, however, they effectively restrict the space of parameter values by assuming positive primary outputs for both firms (i.e. when both prices are set at marginal costs, both firms sell positive outputs). As noted by Amir and Jin (2001), the assumption of positive primary outputs is crucial for the Singh and Vives’s ranking of the equilibrium quantities under the two forms of competition.\textsuperscript{2}

In this paper, we re-consider the comparison of price and quantity competition in the Singh and Vives model allowing for a wider range of cost asymmetry between firms. The analysis reveals that price competition always leads to lower prices and larger social welfare. However, for high degrees of cost asymmetry and low degrees of product differentiation, the efficient firm’s and the industry profits are higher under price than under quantity competition. Therefore, Singh and Vives’s ranking of profits is reversed in a significant region of the relevant parameter space.

The intuition behind our results is as follows. When firms are asymmetric in costs, price competition not only entails lower prices (price effect) but also a stronger selective effect against the market share of the less efficient firm (selection effect) than quantity competition does. While the price effect works towards lower profits under Bertrand than under Cournot competition for both firms, the selection effect works in the opposite direction on the efficient firm’s profits. Moreover, the price effect weakens while the selection effect gets stronger when either the degree of cost asymmetry increases or products are closer substitutes. As a result, the efficient firm earns higher profits under price than under quantity competition when its efficiency advantage over the rival is sufficiently high and products are close substitutes. Moreover, the selection effect entails greater productive efficiency under price than under quantity competition, which explains the reversal of the industry profits’ ranking for high degrees of cost asymmetry and low degrees of product differentiation.

In the homogeneous goods case, the inversion of the ranking of the effi-

\textsuperscript{1}These results hold irrespective of the nature of the goods (substitutes or complements), except that with complementary goods, the ranking of profits is reversed.

\textsuperscript{2}This fact is apparent in the special case of a homogeneous duopoly with linear cost functions, where the assumption of positive primary outputs restricts attention to symmetric costs. With cost asymmetry, the inefficient firm is active in the Cournot equilibrium (provided that the efficiency gap between the two firms is not drastic) but is inactive in the limit-pricing equilibrium arising under Bertrand competition. Therefore, the Singh and Vives’s ranking of the equilibrium quantities clearly does not hold. Amir and Jin (2001) nevertheless maintain the positive primary output assumption, and focus on the case when firms produce a mixture of complementary and substitute goods.
cient firm’s and of the industry profits under the two modes of competition has been noted before by Boone (2001) and Denicolò and Zanchettin (2003), respectively. The present paper generalizes these results showing how the comparison of the equilibrium profits under the two modes of competition depends on both the degree of cost asymmetry and the degree of product differentiation. Further, from the characterization of Bertrand and Cournot equilibria over the entire parameter space of the model we obtain additional results on the behaviour of the equilibrium profits when firms are asymmetric in costs. Namely, under both forms of competition, while the inefficient firm’s profits always decrease as products become closer substitutes, the efficient firm’s profits are non-monotonic in the degree of products differentiation. Therefore, the efficient firm may have a local incentive to reduce the degree of product differentiation. This contrasts with the standard result that arises with symmetric costs, that is, Bertrand and Cournot duopolists always gain from product differentiation (see, among others, Shy (1995), pp. 138-140).

The rest of the paper is organized as follows. Section 2 describes the model, and characterizes Bertrand and Cournot equilibria over the relevant parameter space. Section 3 compares the two forms of competition and presents the main results of the paper. Section 4 collects additional results on the effect of cost asymmetry on firms’ incentive to differentiate products under both forms of competition. Section 5 provides some concluding remarks. All proofs are relegated to an appendix.

2 The model

We consider the following version of the Singh and Vives (1984) model. On the demand side of the market, the representative consumer’s utility is a symmetric-quadratic function of two products, $q_1$ and $q_2$, and a linear function of a numeraire good, $m$,

$$U = \alpha (q_1 + q_2) - \frac{1}{2} (q_1^2 + q_2^2 + 2\gamma q_1 q_2) + m.$$

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3Our results are also related to Hackner (2000). Allowing for both vertical and horizontal product differentiation and symmetric costs, Hackner shows that the “high-quality firms” may earn higher profit with price than with quantity competition when there are more than two competitors in the market. In this paper we show that the Singh and Vives’s ranking of profits is sensitive to cost asymmetry and horizontal differentiation, irrespective of the number of competitors.

4This specification differs from Singh and Vives (1984) in that we assume symmetric demand functions. While this assumption allows us to concentrate on cost asymmetry, it is not restrictive since the relevant measure of firms’ asymmetry under both modes of competition can equally reflect cost and demand asymmetries (see equation 4 below). Without loss of generality, we normalize to one the coefficients of the squared terms in the utility function (i.e. the “own quantity slopes” of the inverse demand functions).
The parameter $\gamma$ measures the degree of product differentiation. We consider the case of substitute goods: $0 \leq \gamma \leq 1$. $\gamma = 0$ and $\gamma = 1$ set the maximum (independent goods) and the minimum (homogeneous goods) degree of differentiation, respectively.

This utility function generates the linear system of inverse demand functions

$$p_i = \alpha - q_i - \gamma q_j\ [i, j = 1, 2; \ i \neq j],$$

whose inversion (by imposing $\gamma < 1$) leads to the direct demand system

$$q_i = \frac{1}{1 - \gamma^2}[(1 - \gamma)\alpha - p_i + \gamma p_j]\ [i, j = 1, 2; \ i \neq j].$$

System (2) gives the direct demand functions provided that prices lead to positive demands for both goods. Discarding the trivial case with zero-demand for both goods, the region of prices where the demand for good $j$ is zero while the demand for good $i$ is positive, $\bar{R}_j$, is identified by

$$\bar{R}_j = \left\{ \begin{array}{lcl} p_i, p_j \geq 0 & (1 - \gamma)\alpha - p_i + \gamma p_j \leq 0 \\ \alpha - p_i > 0 \end{array} \right\}.$$  

Inside region $\bar{R}_j$, the demand function of good $i$ becomes $q_i = \alpha - p_i$.

On the supply side of the market, products $q_1$ and $q_2$ are produced and supplied by firm 1 and firm 2, respectively. Both firms face linear cost functions, but firm 1 is more efficient than firm 2. We set firm 1’s marginal cost at $c_1$, where $0 \leq c_1 < \alpha$ to avoid the trivial case in which neither firm has an incentive to produce. Firm 2’s marginal cost, $c_2$, will lie in the range $c_2 \in [c_1, \alpha]$. We measure the degree of cost asymmetry between the two firms by the ratio:

$$x = \frac{\alpha - c_2}{\alpha - c_1},$$

where $x \in [0, 1]$. As we will see below, this is the relevant measure of firms’ asymmetry in order to characterize the market equilibrium under both Bertrand and Cournot competition. Clearly, $x$ decreases as the efficiency gap between the two firms increases. For $x = 1$ (i.e. $c_2 = c_1$), there is no cost asymmetry. For $x = 0$ (i.e. $c_1 < c_2 = \alpha$), firm 2 is not active in the market irrespective of the form of competition and the degree of products differentiation.

The relevant parameter space. Firm 1 and firm 2 compete either in prices or in quantities. However, the mode of competition matters only in a portion of the parameter space. First, when $\gamma = 0$, both firms are monopolists
on independent segments of the market.\(^5\) Second, for any \(0 < \gamma \leq 1\), if the efficiency gap between the two firms is sufficiently high, then firm 1 can engage in monopoly pricing without bearing any competitive pressure from firm 2, which is driven out of the market irrespective of the form of competition. This is the case when

\[
x \leq x^M(\gamma) = \frac{\gamma}{2}.
\]

Equation (5) identifies an increasing monopoly frontier, \(x^M(\gamma)\), in the space \(S = \{0 \leq \gamma \leq 1; 0 \leq x \leq 1\}\) (see Figure 1). Namely, when products are closer substitutes, a lower efficiency advantage suffices for firm 1 to monopolize the market irrespective of the form of competition.\(^6\)

In our version of the model, the Singh and Vives (1984) assumption of positive primary outputs (i.e. both firms always face positive demand when both prices are set at marginal costs) is binding only for the inefficient firm, and it formally implies the parameter restriction:

\[
x > \gamma.
\]

Therefore, the parameter region considered by Singh and Vives is \(S_{sv} = \{0 < \gamma \leq 1; \gamma \leq x \leq 1\}\). In this paper, we extend the comparison of Bertrand and Cournot equilibria to any relevant combination of cost asymmetry and product differentiation. Since the mode of competition does not matter in the monopoly region (i.e. the region lying below the monopoly frontier in the space \(S\)), the relevant parameter space is given by \(S_r = \{0 < \gamma \leq 1; \gamma \leq x \leq 1\}\). Figure 1 depicts the relevant parameter space \(S_r\) (the area enclosed by the thick contour), the sub-region considered by Singh and Vives (the area above the dotted line), and the resulting extension of the parameter space of the model we consider in this paper (the shaded region).

\(^5\)In this case, we get: \(p_{1M}^i = \frac{\alpha - c_i}{2}, q_{1M}^i = \frac{\alpha - c_i}{2},\) and \(\pi_{1M}^i = \left(\frac{\alpha - c_i}{2}\right)^2\), where \(p_{1M}^i, q_{1M}^i\) and \(\pi_{1M}^i\) are firm \(i\)'s monopoly price, quantity and profits, respectively \((i = 1, 2)\). For \(c_2 > c_1\), we clearly have: \(p_{1M}^i < p_{2M}^i, q_{1M}^i > q_{2M}^i,\) and \(\pi_{1M}^i > \pi_{2M}^i\).

\(^6\)On the contrary, the less efficient firm can never engage in monopoly pricing unless \(\gamma = 0\). Indeed, for \(\gamma \in (0, 1]\) and \(x > x^M(\gamma)\), even firm 2 can price above its marginal cost and face positive demand if the rival prices at \(p_{1M}^i\). Since \(p_{2M}^i \geq p_{1M}^i\) and \(c_2 \geq c_1\), the symmetry of the demand system ensures that the same is true for firm 1.
Cournot competition. Solving the model under Cournot competition, we get: 7

\[ q_C^1 = \frac{\alpha - c_1}{(4 - \gamma)}; \quad \pi_C^1 = \left[ \frac{(\alpha - c_1)(2 - \gamma x)}{(4 - \gamma^2)} \right]^2. \]

\[ q_C^2 = \frac{\alpha - c_2}{(4 - \gamma^2)}; \quad \pi_C^2 = \left[ \frac{(\alpha - c_2)(2 - \gamma x)}{(4 - \gamma^2)} \right]^2. \]

(7)

From equation (7), as \( x \) decreases below 1 (where the equilibrium is symmetric), production, unit-profit and profits increase for firm 1 but decrease for firm 2. However, the inefficient firm remains active in Cournot equilibrium over the entire space \( S_r \) (i.e., given \( \gamma \in (0,1], q_C^2 > 0 \) for any \( x \in (x^M(\gamma), 1] \)).

Bertrand competition. Assume that \( \gamma < 1 \), and suppose that both firms are active in equilibrium.8 Then, Bertrand equilibrium is characterized as follows: 9

\[ \pi_i^B = (\alpha - p_i - \gamma p_j - c_i) x_i \] for any \( x \in (x^M(\gamma), 1] \).

8 Under Bertrand, firm \( i \) chooses \( p_i \) to maximize \( \pi_i = (p_i - c_i \left( \frac{1 - \gamma}{\gamma} \right) \alpha - p_i + \gamma p_j \) \), by taking \( p_j \) as given (i, j = 1, 2; i \neq j). Focusing on an interior equilibrium, the best response
\[ q_1^B = \frac{p_1^B - c_1}{1 - \gamma^2} = \frac{(\alpha - c_1)(2 - \gamma^2 - \gamma x)}{(1 - \gamma^2)(4 - \gamma^2)}; \quad \pi_1^B = \frac{1}{1 - \gamma^2}\left[\frac{(\alpha - c_1)(2 - \gamma^2 - \gamma x)}{4 - \gamma^2}\right]^2; \]

\[ q_2^B = \frac{p_2^B - c_2}{1 - \gamma^2} = \frac{(\alpha - c_1)(2 - \gamma^2)x}{(1 - \gamma^2)(4 - \gamma^2)}; \quad \pi_2^B = \frac{1}{1 - \gamma^2}\left[\frac{(\alpha - c_1)(2 - \gamma^2)x}{4 - \gamma^2}\right]^2. \tag{8} \]

Again, for \( x = 1 \) the equilibrium is symmetric. As \( x \) decreases below 1, production, unit-profit and profits increase for firm 1 but decrease for firm 2. From equation (8), we find that, for any \( \gamma \in (0, 1) \), the inefficient firm is active in Bertrand equilibrium (i.e. \( q_2^B > 0 \)) provided that

\[ x > x^L(\gamma) = \frac{\gamma}{2 - \gamma^2}. \tag{9} \]

Equation (9) defines a limit-pricing frontier, \( x^L(\gamma) \), which is increasing in \( \gamma \), meaning that any efficiency advantage of firm 1 exerts a stronger effect on the rival’s market share when products are closer substitutes (see Figure 1). Moreover, it lies above the monopoly frontier, meaning that, for any degree of products differentiation (but \( \gamma = 0 \)), price competition has a stronger selective effect against the market share of the inefficient firm than quantity competition. Finally, by comparing (6) and (9), it is easy to see that the assumption of positive primary outputs is stronger than the condition for an interior equilibrium under price competition.\(^{10}\)

For \( x^M(\gamma) < x \leq x^L(\gamma) \), the following limit-pricing equilibrium arises under Bertrand competition: \(^{11}\)

\[ p_1^L - c_1 = \frac{1}{\gamma}((\alpha - c_1) (\gamma - x)); \quad q_1^L = \frac{1}{\gamma}(\alpha - c_1) x; \]

\[ \pi_1^L = \frac{1}{\gamma^2}\left[\frac{(\alpha - c_1)^2}{4}(\gamma - x)x\right]; \]

\[ p_2^L - c_2 = q_2^L = \pi_2^L = 0. \tag{10} \]

functions are: \( p_i = \frac{1}{\gamma}\left[1 - \gamma\alpha + c_i + \gamma p_j\right], (i, j = 1, 2; i \neq j) \). Equation (8) follows from the best response functions above and equation (4).

\(^{10}\)Indeed, condition (6) identifies a lower bound for \( x \) which lies above the limit-pricing frontier for any \( \gamma \in (0, 1) \) (i.e. the dotted line in Figure 1).

\(^{11}\)That is, the efficient firm prices good 1 at the maximum level leaving the rival zero-demand for any price of good 2 higher than the rival’s marginal cost. Formally, given \( \gamma \) and \( x < 1 \), the best response function of the efficient firm follows the expression given in footnote 9 until prices reach the boundary of the region \( \mathcal{R}_2 \), where the demand for good 2 is zero. It kinks thereafter and continues along the boundary, i.e., from eq. (3), \( p_1 = \frac{1}{\gamma}|p_2 - (1 - \gamma)\alpha| \). Moreover, the best response function of the inefficient firm, \( p_2 = \frac{1}{\gamma}[1 - \gamma\alpha + c_2 + \gamma p_1] \), shifts outwards as \( c_2 \) increases (\( x \) decreases). When the efficiency gap is sufficiently high (i.e. \( x \leq x^L(\gamma) \)), it crosses the best response function of the efficient firm along the boundary of region \( \mathcal{R}_2 \). Equation (10) follows from the best response functions above and equation (4).
3 Comparison of Bertrand and Cournot equilibria

Let us denote with $S_A$ the region of the relevant parameter space where both firms are active under Bertrand competition (the area above the limit-pricing frontier in Figure 1), and with $S_L$ the region where Bertrand competition entails a limit-pricing equilibrium (the area between the limit-pricing and the monopoly frontiers). Singh and Vives (1984) restrict their attention to a portion of region $S_A$. In this section we extend the comparison of Bertrand and Cournot equilibria over the entire space $S_r = S_A + S_L$. We start by comparing the equilibrium prices and quantities.

**Lemma 1 (prices)** The equilibrium prices of both firms are higher under Cournot than under Bertrand competition over the entire space $S_r$.

**Lemma 2 (quantities)** The efficient firm produces more under Bertrand than under Cournot competition over the entire space $S_r$. For the inefficient firm, Bertrand production exceeds Cournot production iff $x > \gamma$ (i.e. the same as condition (6)).

Lemma 1 confirms the Singh and Vives's ranking of Bertrand and Cournot equilibrium prices over the entire space $S_r$. Namely, under price competition firms perceive a more elastic demand than under quantity competition, and this refrains them from increasing prices. Lemma 2 emphasizes the more selective effect of price competition against the market share of the inefficient firm. While the efficient firm always produces more under Bertrand competition, the inefficient firm produces less under Bertrand than under Cournot competition when the cost asymmetry is sufficiently strong and/or products are close substitutes, inside region $S_A$, as well as over the entire region $S_L$.\(^{12}\)

Turning to the comparison of the equilibrium profits, we first show that the Singh and Vives’s ranking holds over the entire region $S_A$.

**Proposition 1 (Singh and Vives 1984)** If both firms are active under Bertrand competition, they both earn higher profits under Cournot than under Bertrand competition.

Consider now region $S_L$. Obviously, in this region the inefficient firm’s profits are always higher with quantity competition (i.e. positive rather than zero). On the contrary, we prove now that the efficient firm’s and the industry profits are higher under Bertrand than under Cournot competition in a significant portion of region $S_L$.

\(^{12}\)More precisely, Lemma 2 shows that the portion of region $S_A$ where both firms produce more under Bertrand than under Cournot coincides with the parameter region considered by Singh and Vives (i.e. $S_{sv}$ in Figure 1).
Proposition 2 (efficient firm’s profits) For any degree of product differentiation (but \(\gamma = 0\)), there exists a critical level of cost asymmetry inside region \(S_L\), \(\hat{x}(\gamma) = \frac{4\gamma}{8-4\gamma+\gamma^2}\), such that the efficient firm’s profits are higher (resp. lower) with Bertrand than with Cournot competition when \(x < \hat{x}(\gamma)\) (resp. \(x > \hat{x}(\gamma)\)).

Proposition 3 (industry profits) For any degree of product differentiation (but \(\gamma = 0\)), there exists a critical level of cost asymmetry inside region \(S_L\), \(\bar{x}(\gamma) = \frac{\gamma(4+\gamma^2)}{8-4\gamma+\gamma^2}\), such that industry profits are higher (resp. lower) with Bertrand than with Cournot competition when \(x < \bar{x}(\gamma)\) (resp. \(x > \bar{x}(\gamma)\)).

Figure 3 shows the two loci, \(\hat{x}(\gamma)\) and \(\bar{x}(\gamma)\), and the portions of region \(S_L\) where the efficient firm’s and the industry profits are higher with Bertrand competition.

The interpretation of these results relies on the comparison of the effects on prices and market shares exerted by the two forms of competition over regions \(S_A\) and \(S_L\). On one hand, both firms face lower prices with Bertrand competition over the relevant space of the model, and this tends to make their profits lower under Bertrand than under Cournot competition (price effect). On the other hand, when firms are asymmetric in costs,

\[\hat{x}(\gamma) = \frac{4\gamma}{8-4\gamma+\gamma^2}\]

\[\bar{x}(\gamma) = \frac{\gamma(4+\gamma^2)}{8-4\gamma+\gamma^2}\]

Figure 3

\(^{13}\)Notice that the critical levels \(\hat{x}(\gamma)\) (Proposition 2) and \(\bar{x}(\gamma)\) (Proposition 3) are both monotonically increasing in \(\gamma\), taking values \(\hat{x}(0) = \bar{x}(0) = 0\) and \(\hat{x}(1) < \bar{x}(1) < 1\). Therefore, by Proposition 2, for any \(x < \hat{x}(1)\), we can always identify a critical degree of product differentiation inside region \(S_L\), namely \(\hat{\gamma}(x) = \hat{x}^{-1}(\gamma)\), such that the efficient firm earns higher profit under Bertrand for \(\gamma > \hat{\gamma}(x)\). Similarly, by Proposition 3, given any \(x < \bar{x}(1)\), \(\bar{\gamma}(x) = \bar{x}^{-1}(\gamma)\) sets a critical degree of product differentiation inside region \(S_L\), such that industry profits are higher under Bertrand for \(\gamma > \bar{\gamma}(x)\).
price competition tends to reduce more the market share of the less efficient firm than quantity competition does. Obviously this tends to make the efficient firm’s profits higher, and the inefficient firm’s profits lower, under Bertrand than under Cournot competition (selection effect). Both effects work in the same direction for the inefficient firm, while they operate in opposite directions for the efficient firm. Moreover, the price effect weakens, whilst the selection effect gets stronger, when either the efficiency gap between the two firms increases or the degree of product differentiation decreases.\(^{14}\) Accordingly, the price effect dominates the selection effect (i.e. the efficient firm gets higher profits under Cournot) for mild degrees of cost asymmetry and/or strongly differentiated products (Proposition 1), whilst the selection effect prevails on the price effect (i.e. the efficient firm earns higher profits under Bertrand) when the degree of cost asymmetry is sufficiently high and/or products are close substitutes (Proposition 2). Finally, while the price effect works towards lower industry profits under Bertrand than under Cournot, the stronger selective mode of price competition relocates production from the inefficient to the efficient firm, working towards higher industry profits under Bertrand (productive efficiency effect). The productive efficiency effect dominates the price effect (i.e. industry profits are higher under Bertrand) for high degrees of cost asymmetry and/or low degrees of product differentiation (Proposition 3).

The dominance of the selection and productive-efficiency effects over the price effect close to the monopoly frontier can be easily perceived by the following argument. Consider a slight reduction of the efficiency gap (i.e. a small increase in $x$) or a slight increase in product differentiation (i.e. a small decrease in $\gamma$) starting from the monopoly frontier. Under Bertrand competition, the efficient firm is now forced by outside competition to price below the monopoly level. However, since the equilibrium price falls just below the monopoly price, the negative effect on its profits is only second order. In contrast, the inefficient firm holds a (slightly) positive market share under Cournot competition. Whilst the efficient firm’s price decreases less than under Bertrand, the reduction in the market share of the efficient firm causes a first order negative effect on its profits. As a consequence, the efficient firm’s profits are greater under Bertrand. Furthermore, being close to the monopoly frontier, the inefficient firm is forced to price at a level almost equal to its marginal cost under Cournot, so that the positive effect on its profits is second order as well. Therefore, industry profits are also greater under Bertrand.

\(^{14}\)The selection effect reaches its maximum intensity on the limit-pricing frontier, where the market share of the inefficient firm is zero under Bertrand but still positive under Cournot. On the other hand, the price effect vanishes on the monopoly frontier, where the prices are identical under the two modes of competition (i.e., the monopoly price, $p^M_1$, for the efficient firm, and the marginal cost, $c_2$, for the inefficient firm).
We conclude this section with the welfare comparison of Bertrand and Cournot equilibria over the relevant space $S_r$. From the utility function, total surplus equals

$$TS = (\alpha - c_1)q_1 + (\alpha - c_2)q_2 - \left[ \frac{1}{2} (q_1 + q_2)^2 - (1 - \gamma)q_1q_2 \right], \quad (11)$$

while consumer surplus is given by the difference between total surplus and industry profits. Clearly, total surplus increases with quantities (as far as the marginal utility of both goods exceeds the respective marginal cost), while consumer surplus always decreases with prices. As shown by Lemma 2, Singh and Vives restrict attention to the portion of region $S_A$ where both firms produce more under Bertrand than under Cournot competition. Since prices are lower under Bertrand, they can easily conclude that both total surplus and consumer surplus are larger under Bertrand competition. Looking at the other portions of the relevant space, since prices are lower under Bertrand everywhere (see Lemma 1), the Singh and Vives’s ranking of consumer surplus holds over the entire space $S_r$. This implies immediately that total surplus is also larger under Bertrand over the portion of region $S_L$ where price competition entails higher industry profits (see Proposition 3). However, the ranking of total surplus is not immediate either over the portion of region $S_A$ where the inefficient firm produces more under Cournot, or over the portion of region $S_L$ where industry profits are higher under Cournot. Nevertheless, the following proposition shows that the Singh and Vives’s ranking of total surplus extends over the entire space $S_r$.

**Proposition 4 (welfare)** Total surplus is higher under Bertrand than under Cournot competition over the entire space $S_r$.

Summarizing, while consumers always gain from a switch from Cournot to Bertrand competition, for low degrees of cost asymmetry (and/or high degrees of product differentiation) both firms lose since the price effect dominates the productive efficiency effect. The total surplus rises because the increase in consumer surplus exceeds the decrease in industry profits. In contrast, when the cost asymmetry is high (and/or products are close substitutes), a switch from Cournot to Bertrand competition increases both consumer surplus and industry profits, since now the overall effect on industry profits reflects the dominance of the productive efficiency effect over the price effect.

\footnote{Notice that, with imperfect substitute goods and asymmetric costs, the comparison of equilibrium prices is not conclusive for the ranking of total surplus, since lower equilibrium prices do not necessarily imply higher equilibrium quantities for both goods.}
4 Product differentiation and the market leader’s profit

Our characterization of Bertrand and Cournot equilibria over the entire space \( S \) reveals immediately that, in the presence of cost asymmetry, the efficient firm’s profits may be non-monotonic in the degree of product differentiation under both forms of competition. In fact, for sufficiently high degrees of cost asymmetry (i.e. low values of \( x \)), the efficient firm earns the monopoly profits either for \( \gamma = 0 \) or along the monopoly frontier. Since the profit functions are continuous under both forms of competition, in both cases the efficient firm’s profits must be non-monotonic in \( \gamma \) for high levels of cost asymmetry. In contrast, the standard result with symmetric costs is that, under both forms of competition, profits always decrease as products become less differentiated (see Shy (1995), pp. 138-140).

The following two lemma provide a complete characterization of the behavior of firms’ equilibrium profits with respect to the degree of product differentiation, under the two forms of competition.

**Lemma 3** Under Cournot competition, for any \( x \in (0, 1] \), the inefficient firm’s profits always decrease as \( \gamma \) increases from 0 to \( \min\{\gamma^M(x), 1\} \), where \( \gamma^M(x) \equiv x^{M-1}(\gamma) \). On the contrary, there exist a threshold level \( x^C \in (x^M(1), 1) \) and a critical locus \( \gamma^C(x) \in (0, \min\{\gamma^M(x), 1\}) \), such that the efficient firm’s profits increase with \( \gamma \) if \( x < x^C \) and \( \gamma > \gamma^C(x) \), and decrease with \( \gamma \) otherwise.

**Lemma 4** Under Bertrand competition, for any \( x \in (0, 1] \), the inefficient firm’s profits always decrease as \( \gamma \) increases from 0 to \( \gamma^L(x) \equiv x^{L-1}(\gamma) \). On the contrary, for any \( x \in (0, 1) \), there exists a critical value \( \gamma^B(x) \in (0, \gamma^L(x)) \), such that the efficient firm’s profits decrease with \( \gamma \) from 0 to \( \gamma^B(x) \), while they increase with \( \gamma \) from \( \gamma^B(x) \) to \( \min\{\gamma^M(x), 1\} \).

As a consequence of Lemma 3 and 4, the conventional result that Bertrand and Cournot duopolists always gain from product differentiation should be amended in the presence of cost asymmetries. Indeed, when the degree of cost asymmetry is sufficiently high and/or products are initially close substitutes, the efficient firm may have a local incentive to reduce the degree of product differentiation. This is more likely to happen with Bertrand than with Cournot competition, since, under Bertrand, the efficient firm’s profits are non-monotonic in \( \gamma \) for any positive degree of cost asymmetry (Lemma 4), while a sufficient degree of cost asymmetry is required under Cournot (Lemma 3).

The intuition behind these results is as follows. A higher degree of product substitutability lowers the demand functions of both goods, as consumers value less any bundle of the two products in terms of the numeraire good. If
firms are symmetric in costs, this leads to a new symmetric equilibrium in which firms earn lower profits under both forms of competition. However, in the presence of cost asymmetry, the demand function of the inefficient firm is shifted down more since consumers substitute the higher priced good for the lower priced one. Under Cournot competition the efficient firm benefits from the larger reduction of the rival’s production, which counteracts the negative effect exerted by the increase in $\gamma$ on its inverse demand function. For sufficiently high degrees of cost asymmetry, the positive effect will eventually prevail when products become sufficiently close substitutes. Under Bertrand competition, the larger decrease in the rival’s demand function allows the efficient firm to capture more of the rival’s demand at any given price. This counteracts the negative effect due to the adverse shift in its demand function. Now, for any degree of cost asymmetry, the positive effect will eventually prevail when products are sufficiently close substitutes.

5 Conclusions

In this paper we have re-considered the comparison of Bertrand and Cournot competition within the standard model of a horizontally differentiated duopoly with linear demand and cost functions. Our main innovation with respect to Singh and Vives (1984) has been to enlarge the parameter space by allowing for any relevant combination of cost asymmetry and product differentiation. Our main result, that the efficient firm’s and industry profits are higher under Bertrand competition for high degrees of cost asymmetry and/or low degrees of product differentiation, contrasts with the Singh and Vives’s ranking of the equilibrium profits under the two forms of competition.

The intuition for this result relies on the stronger selective mode of price competition (selection effect), which also entails greater productive efficiency (productive efficiency effect). The selection and the productive efficiency effects work towards higher profits for the efficient firm and for the industry under Bertrand than under Cournot competition, contrasting the opposite effect due to the lower equilibrium prices arising under Bertrand (price effect). With high degrees of cost asymmetry and/or low degrees of product differentiation, the selection and the productive efficiency effects dominate the price effect, inverting the standard ranking of profits under the two forms of competition.

We have also shown that, because of cost asymmetry, the efficient firm’s profits may be non-monotonic in the degree of product differentiation under both modes of competition, so that a local incentive towards less differentiation may arise.

The price, selection and productive efficiency effects are well known in

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16 By equation (2), given an initial pair of prices, the negative shift in the demand function of each firm is inversely related to the rival’s price.
the IO literature that analyses the intensity of market competition and its effect on firms’ incentive to innovate.\footnote{Among others, Delbono and Denicolò (1990); Bester and Petrakis (1993); Qiu (1998); Aghion and Shankerman (2000); Boone (2000).} Our paper contributes to this literature by offering a complete map of the composition of these effects along the two dimensions of cost asymmetry and horizontal products differentiation, within a model which is widely employed in this field. The inversion of the profits ranking between the two modes of competition, as well as the efficient firm’s local incentive towards less product differentiation, are significant results for multi-stage models with product or process innovation at the early stages. More in general, our results may find interesting applications in many other strands of the literature which employ the model as a “building block”, such as games of tacit collusion under the two forms of competition (e.g. Deneckere (1983), Majerus (1988), Lamberti (1997)), or models which endogenize the timing and the strategic variable of competition in oligopoly games (e.g. Singh and Vives (1984), Hamilton and Slutsky (1990)).

Appendix

Proof of Lemma 1. Using equations (7) and (8), over region $S_A$ we get:

$$p_C^1 - p_B^1 = \frac{\gamma^2}{4-\gamma} (\alpha - c_1)$$

$$p_C^2 - p_B^2 = \frac{\gamma^2}{4-\gamma} (\alpha - c_1) x.$$

Both expressions are positive for $\gamma, x \in (0, 1]$ (and hence, everywhere over region $S_A$). Turning to region $S_L$, for the inefficient firm we clearly have $p_C^2 > p_L^2 = c_2$. For the efficient firm, equations (7) and (10) yield:

$$p_C^1 - p_L^1 = \frac{(2 - \gamma^2)(\alpha - c_1)}{\gamma (4 - \gamma^2)} [2x - \gamma].$$

This expression is positive for $\gamma, x \in (0, 1)$.

Proof of Lemma 2. From equations (7) and (8), over region $S_A$ we get:

$$q_B^1 - q_C^1 = \frac{\gamma^2(\alpha - c_1)}{(1-\gamma)(4-\gamma)} [1 - \gamma x]$$

$$q_B^2 - q_C^2 = \frac{\gamma^2(\alpha - c_1)}{(1-\gamma)(4-\gamma)} [x - \gamma].$$

The first expression is positive for $\gamma, x \in (0, 1)$. From the second expression we find immediately that, for any $\gamma \in (0, 1)$, $q_B^2 - q_C^2 \geq 0 \iff x \geq \frac{\gamma^2}{\gamma}$. For
$x = 1$ and $\gamma \to 1$, the model approaches a standard homogeneous duopoly with linear-symmetric cost functions (i.e. $c_2 = c_1$), and both expressions converge to the positive limit $\frac{x^2(\alpha - c_2)}{(1 - \gamma)(1 - \gamma^2)}$. Turning to region $S_L$, for the inefficient firm we clearly have $q^C_L > q^L_L = 0$. For the efficient firm, equations (7) and (10) yield:

$$q^L_1 - q^C_1 = \frac{2(\alpha - c_1)}{\gamma (4 - \gamma^2)} (2x - \gamma).$$

This expression is positive for $\gamma \in (0, 1)$ and $x > \frac{\gamma}{2} = x^M(\gamma)$ (that is, everywhere over region $S_L$).

**Proof of Proposition 1.** Notice that $\gamma, x \in (0, 1)$ over region $S_A$, setting aside point $(x = 1, \gamma = 1)$ where products are homogeneous and firms are symmetric. Clearly, in that point both firms earn higher profits under Cournot.

Consider first firm 1. Using equations (7) and (8), $\pi^C_1 \geq \pi^B_1$ leads to:

$$\gamma^3 [\gamma x^2 - 2x + \gamma] \leq 0.$$

Given any $\gamma \in (0, 1)$, this inequality holds for $x \in [\bar{x}(\gamma), \bar{\pi}(\gamma)]$, where $\bar{x}(\gamma) = \frac{1}{\gamma}(1 - \sqrt{1 - \gamma^2})$ and $\bar{\pi}(\gamma) = \frac{1}{\gamma}(1 + \sqrt{1 - \gamma^2})$. Now, $\bar{\pi}(\gamma) > 1$ for $\gamma \in (0, 1)$. Moreover, $\bar{x}(\gamma)$ lies below the limit-pricing frontier. Indeed, $\bar{x}(\gamma) < x^L(\gamma)$ leads to the inequality $\sqrt{1 - \gamma^2}(1 - \sqrt{1 - \gamma^2})^2 > 0$, which is satisfied for any $\gamma \in (0, 1)$. Hence, $S_A \subset [\bar{x}(\gamma), \bar{\pi}(\gamma)]$, so that $\pi^C_1 > \pi^B_1$ everywhere over region $S_A$.

Turning to firm 2, from equations (7) and (8) we find that $\pi^C_2 \geq \pi^B_2$ leads exactly to the same inequality $\gamma^3 [\gamma x^2 - 2x + \gamma] \leq 0$.

**Proof of Proposition 2.** Notice first that $\gamma, x \in (0, 1)$ over region $S_L$. Using equations (7) and (10), $\pi^L_1 \geq \pi^C_1$ is equivalent to

$$(16 - 8\gamma^2 + 2\gamma^4) x^2 - \gamma (16 - 4\gamma^2 + \gamma^4) x + 4\gamma^2 \leq 0.$$

Given any $\gamma \in (0, 1)$, this inequality is satisfied for $x \in [x^M(\gamma), \bar{x}(\gamma)]$, where $x^M(\gamma) = \frac{\gamma}{2}$ is the monopoly frontier, and $\bar{x}(\gamma) = \frac{\gamma}{2 - \gamma^2 + \gamma^2}$.

Since $x^M(\gamma) < \bar{x}(\gamma) < x^L(\gamma) (= \frac{\gamma}{2 - \gamma^2})$ for any $\gamma \in (0, 1]$, the critical level $\bar{x}(\gamma)$ always lies inside region $S_L$. It is also easy to verify that $\bar{x}(\gamma)$ monotonically increases with $\gamma$.

**Proof of Proposition 3.** From equations (7) and (10), Bertrand and Cournot industry profits over region $S_L$ are, respectively:

$$\pi^L = \frac{(\alpha-c_1)^2}{\gamma} [\gamma x - x^2]$$

$$\pi^C = \frac{(\alpha-c_1)^2}{(4 - \gamma^2)^2} \left[ x^2 (4 + \gamma^2) - 8\gamma x + (4 + \gamma^2) \right].$$

Using these expressions, $\pi^L \geq \pi^C$ is equivalent to

$$[16 - 4\gamma^2 + 2\gamma^4] x^2 - \gamma (16 + \gamma^4) x + \gamma^2 (4 + \gamma^2) \leq 0.$$
Solving in $x$, we find this inequality satisfied for $x \in [x^M(\gamma), \tilde{x}(\gamma)]$, where $x^M(\gamma) = \frac{3}{2}$ is the monopoly frontier, and $\tilde{x}(\gamma) = \frac{\gamma(4 + \gamma^2)}{2 + \gamma^2}$.

It is easy to verify that $x^M(\gamma) < \tilde{x}(\gamma) < x^L(\gamma)$ for any $\gamma \in (0, 1]$, so that $\tilde{x}(\gamma)$ always lies inside region $S_L$. Also, $\tilde{x}(\gamma)$ is monotonically increasing in $\gamma$.

**Proof of Proposition 4.** Consider first region $S_A$. Notice that $\gamma, x \in (0, 1)$ over region $S_A$, setting aside point $(x = 1, \gamma = 1)$. In this point, products are homogeneous and firms are symmetric, and the welfare comparison between Bertrand and Cournot equilibria is standard.

From equations (7) and (11), the total surplus in Cournot equilibrium is:

$$TS^C = \left(\frac{a - c}{\delta^2}\right)^2 \left[(6 - \frac{1}{2}\gamma^2)x^2 - (8\gamma - \gamma^3)x + (6 - \frac{1}{2}\gamma^2)\right].$$

Similarly, from equations (8) and (11), in Bertrand equilibrium we have

$$TS^B = \left(\frac{a - c}{(1 - \gamma^2)(4 - \gamma^2)}\right)^2 \left[(6 - \frac{21}{2}\gamma^2 + \frac{11\gamma^4 - \gamma^6}{2})x^2 - (8\gamma + 3\gamma^5 - 11\gamma^3)x + (6 - \frac{21}{2}\gamma^2 + \frac{11\gamma^4 - \gamma^6}{2})\right].$$

Imposing $TS^B > TS^C$, implies

$$(2\gamma^2 - \frac{3\gamma^4}{2} - \frac{\gamma^6}{2})x^2 - (6\gamma^3 - 7\gamma^5 + \gamma^7)x + (2\gamma^2 - \frac{3\gamma^4}{2} - \frac{\gamma^6}{2}) > 0.$$

The discriminant of the inequality above,

$$\Delta = \gamma^4(60\gamma^2 + 55\gamma^6 - 85\gamma^4 + \gamma^{10} - 15\gamma^8 - 16),$$

is negative for any $\gamma \in (0, 1)$, whilst $2\gamma^2 - \frac{3\gamma^4}{2} - \frac{\gamma^6}{2} > 0$. This suffices to prove that the inequality is always satisfied for $\gamma, x \in (0, 1)$.

Consider now region $S_L$. The total surplus under Cournot competition is still given by the expression above. Under Bertrand competition, equations (10) and (11) yield

$$TS^L = \frac{(a - c)^2}{\gamma^2} \left[\gamma x - \frac{1}{2}x^2\right].$$

Imposing $TS^L \geq TS^C$, implies

$$-(8 + 2\gamma^2)x^2 + 16\gamma x - \frac{1}{2}\gamma^2(12 - \gamma^2) \geq 0.$$

Given any $\gamma \in (0, 1]$, this inequality is satisfied for $x \in [x^M(\gamma), x^{TS}(\gamma)]$, where $x^M(\gamma) = \frac{\gamma}{2}$ is the monopoly frontier, and $x^{TS}(\gamma) = \frac{\sqrt{12 - \gamma^2}}{3 + \gamma^2}$. It is easy to verify that $x^{TS}(\gamma) > x^L(\gamma) (= \frac{\sqrt{12 - \gamma^2}}{3 + \gamma^2})$. This means that $S_L \subset [x^M(\gamma), x^{TS}(\gamma)]$, so that $TS^L > TS^C$ over region $S_L$.

**Proof of Lemma 3.** From equation (7), we get:

$$\frac{\partial \pi^C}{\partial \gamma} = \frac{2(a - c)\sqrt{\delta^2}}{(4 - \gamma^2)^{3/2}} \left[-(4 + \gamma^2) + 4x\gamma\right],$$

$$\frac{\partial \pi^C}{\partial \gamma} = \frac{2(a - c)\sqrt{\delta^2}}{(4 - \gamma^2)^{3/2}} \left[-x(4 + \gamma^2) + 4\gamma\right].$$
From the first expression, \( \frac{\partial \pi^C}{\partial \gamma} \) is strictly negative as far as \( \pi^C_2 > 0 \) (it equals zero on the monopoly frontier, where \( \pi^C_2 = 0 \)). From the second expression, \( \frac{\partial \pi^C}{\partial \gamma} \leq 0 \) is equivalent to \( x \leq \frac{4\gamma}{4+\gamma} \equiv x^C(\gamma) \). It is easy to verify that \( x^C(\gamma) \) is monotonically increasing in \( \gamma \), taking values \( x^C(0) = 0 \) and \( x^C(1) = \frac{4}{5} < 1 \). Moreover, \( x^C(\gamma) > x^M(\gamma) = \frac{\gamma}{2} \) for any \( \gamma \in (0, 1) \) (i.e. \( x^C(\gamma) \) always lies inside the relevant space \( S_r \)). This suffices to prove that, for any \( x \in (0, \frac{\gamma}{2}) \), the efficient firm’s profits decrease as \( \gamma \) rises from 0 to \( \gamma^C(x) = x^{C-1}(\gamma) \), while they increase as \( \gamma \) rises from \( \gamma^C(x) \) to the minimum between 1 and \( \gamma^M(x) = x^{M-1}(\gamma) \).

**Proof of Lemma 4.** We start with firm 1’s profits over region \( S_A \). From equation (8), we get:

\[
\frac{\partial \pi^B}{\partial \gamma} = \frac{2(\gamma-\gamma^2-\gamma^3)}{(4-\gamma^2)(1-\gamma^2)} \left[ \gamma \left( 4 - 2\gamma^2 + \gamma^4 \right) - x \left( 4 + \gamma^2 - 2\gamma^4 \right) \right].
\]

The first term of this expression is strictly positive for \( x, \gamma \in (0, 1) \). Then, \( \frac{\partial \pi^B}{\partial \gamma} \geq 0 \) is equivalent to \( x \leq \frac{\gamma (4-2\gamma^2+\gamma^4)}{(4+\gamma^2)(1-\gamma^2)} \equiv x^B(\gamma) \). It is easy to verify that \( x^B(\gamma) \) is monotonically increasing in \( \gamma \), taking the values \( x^B(0) = 0 \) and \( x^B(1) = 1 \). Moreover, \( x^B(\gamma) > x^L(\gamma) = \frac{\gamma}{2-\gamma} \) for any \( \gamma \in (0, 1) \) (i.e. \( x^B(\gamma) \) always lies inside region \( S_A \)). This suffices to prove that, for any \( x \in (0, 1) \), the efficient firm’s profits decrease as \( \gamma \) rises from 0 to \( \gamma^B(x) = x^{B-1}(\gamma) \), while they increase as \( \gamma \) rises from \( \gamma^B(x) \) to \( \gamma^L(x) = x^{L-1}(\gamma) \).

Turning to region \( S_L \), from equation (10) we obtain:

\[
\frac{\partial \pi^L}{\partial \gamma} = \frac{(\gamma-\gamma^2)}{\gamma^2} [x(2x-\gamma)],
\]

which is positive for any \( x > x^M(\gamma) = \frac{\gamma}{2} \) (that is, everywhere over region \( S_L \)).

Consider now firm 2’s profits over region \( S_A \). From equation (8), we get

\[
\frac{\partial \pi^B}{\partial \gamma} = \frac{2(\gamma-\gamma^2)}{(4-\gamma^2)(1-\gamma^2)} \left[ x\gamma \left( 4 - 2\gamma^2 + \gamma^4 \right) - \left( 4 + \gamma^2 - 2\gamma^4 \right) \right].
\]

The first term of this expression is positive for \( 1 \leq x < x^L(\gamma) = \frac{\gamma}{1-\gamma^2} \) (i.e. over region \( S_A \)). Then, \( \frac{\partial \pi^B}{\partial \gamma} \leq 0 \) is equivalent to \( x \leq \frac{(4+\gamma^2-2\gamma^4)}{\gamma(4-2\gamma^2+\gamma^4)} = \frac{1}{x^2(\gamma)} \). Since \( x^B(\gamma) \in [0, 1] \), its reciprocal is always greater than 1 (it equals 1 only for \( \gamma = 1 \)), and the inequality above is always satisfied.

**References**


