The Max-Convolution Approach to Equilibrium Models with Indivisibilities [1]

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Abstract: This paper studies a competitive market model for trading indivisible commodities. Commodities can be desirable or undesirable. Agents' preferences depend on the bundle of commodities and the quantity of money they hold. We assume that agents have quasi-linear utilities in money. Using the max-convolution approach, we demonstrate that the market has a Walrasian equilibrium if and only if the potential market value function is concave with respect to the total initial endowment of commodities. We then identify sufficient conditions on each individual agent's behavior. In particular, we introduce a class of new utility functions, called the class of max-convolution concavity preservable utility functions. This class of utility functions covers both the class of functions which satisfy the gross substitutes condition of Kelso and Crawford (1982), or the single improvement condition, or the no complementarities condition of Gul and Stacchetti (1999), and the class of $M^\natural$-concave functions of Murota and Shioura (1999).

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1 Introduction

In this paper we study a competitive market model with a finite number of agents for trading various indivisible commodities. Commodities can be desirable such as houses or cars, or undesirable such as aging nuclear plants. Each agent is initially endowed with several units of each commodity and some amount of money. Agents' preferences depend on the bundle of commodities and the quantity of money they hold. As most of the recent literature does, we also focus on a particular but important case in which all agents have quasi-linear utilities in money. This model is fairly broad. Its related examples include the models in Bikhchandani and Mamer (1997), Ma (1998), Bevia, Quinzii and Silva (1999), Gul and Stacchett (1999), and the assignment models in Koopmans and Beckman (1957), and Shapley and Shubik (1972). It should be aware that the models in Kelso and Crawford (1982), Laan, Talman and Yang (1997, 2002), Yang (2000), Danilov, Koshevoy and Murota (2001), Fujishige and Yang (2002), allow for somehow more general situations in the sense that quasi-linearity in money is not required.

It is well-known that there exists a Walrasian equilibrium under rather mild conditions in any model in which every agent demands only one indivisible object but has preferences over different objects. See e.g., Quinzii (1984), and Kaneko and Yamamoto (1986). Unfortunately, the existence of a Walrasian equilibrium is not guaranteed anymore even under many familiar standard conditions if agents are allowed to demand more than one indivisible object. In a seminal article, Kelso and Crawford (1982) introduce the gross substitutes (GS) condition for the existence of a nonempty core (and equilibrium) in a fairly general two-sided matching model with money. This condition has become a benchmark condition for the existence of equilibrium in matching, equilibrium, and auction models where agents are allowed to demand as many indivisible objects as they wish. Gul and Stacchetti (1999) present two new and interesting alternative conditions, the single improvement (SI), and no complementarities (NC) conditions, and have shown that these conditions are equivalent to the gross substitutes condition. Nevertheless, the GS, SI and NC conditions are

not conditions on the primitive characteristics of the economy (the utility functions) but conditions on the derived demand correspondences. This raises a natural question. What kind of functions satisfy the GS, or equivalently, SI, or NC condition? Three special classes of functions satisfying the GS conditions are given by Kelso and Crawford (1982), Bevia, Quinzii and Silva (1999), and Gul and Stacchett (1999).

In an apparently unrelated development, Murota (1998, 2003) and Murota and Shioura (1999) have recently developed an interesting theory of discrete convex analysis in the field of discrete optimization. This theory could play an important role in solving problems of efficient allocation of indivisible resources. Danilov, Koshevoy and Murota (2001), and Fujishige and Yang (2002) have applied this theory to the equilibrium models with indivisibilities and established the existence of equilibrium. Fujishige and Yang (2003) have shown that a utility function satisfies the GS condition if and only if it is an $M^\natural$-concave function introduced by Murota and Shioura, and thus bridged the gap between the two quite different identities, the GS condition and the $M^\natural$-concave functions. Subsequent to Fujishige and Yang (2003), Danilov, Koshevoy and Lang (2003), Murota and Tamura (2003) have independently shown that the GS, SI, and NC conditions and their relation with $M^\natural$-concave functions can be analogously extended to more general situations.

In this paper we demonstrate through the max-convolution approach that the market has a Walrasian equilibrium if and only if the potential market value function is concave with respect to the total initial endowment of commodities. We then identify sufficient conditions on each individual agent's behavior. In particular, we introduce a class of new utility functions, called the class of max-convolution concavity preservable utility functions. This class of utility functions covers both the class of functions which satisfy the gross substitutes condition of Kelso and Crawford (1982), or the single improvement condition, or the no complementarities condition of Gul and Stacchetti (1999), and the class of $M^\natural$-concave functions of Murota and Shioura (1999). Compared with the existing approaches, the approach provided here has some advantages: First, it enables us to establish a very natural and intimate relationship between equilibrium and concavity and also helps us better understand what are the fundamental differences between the indivisible goods market and the divisible goods market in term of existence conditions. Second, its ar-

gument is more transparent and it also allows us to derive the existing existence results, including the well-known gross substitutes condition of Kelso and Crawford (1982), from a unifying perspective. Third, this approach leads to a natural application of the discrete concave functions introduced by Murota (1998, 2003), Murota and Shioura (1999), and also indicates a new way of generating more general utility functions for the existence of equilibrium.

This paper is organized as follows. In Section 2 we introduce the market model. In Section 3 we establish two necessary and sufficient conditions for the existence of an equilibrium in the model. In Section 4 we identify sufficient conditions on the behavior of each individual agent and also make a comparison of the indivisible goods market with the divisible goods market.

2 The Market Model

First, we introduce some notation. The set I_k denotes the set of the first k positive integers. The set $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$. The vector $\mathbf{0}$ denotes the vector of zeros. The vector $e(i)$, $i \in I_n$, is the ith unit vector of $\mathbb{R}^n$. Furthermore, $x \cdot y$ means the inner product of vectors x and y.

Consider a market for trading various indivisible commodities. In the market there are m agents, n indivisible commodities, and money. The set of all agents will be denoted by $T = \{1, 2, \cdots, m\}$. Each agent i is initially endowed with a bundle $\omega^i \in \mathbb{Z}^n_+$ of goods and some amount m_i of money. Let ω stand for the total initial endowment of indivisible commodities in the market, i.e., $\omega = \sum_{i \in T} \omega^i$. Thus, for each commodity $h = 1, \cdots, n$, there are ω_h units available in the market. It is understood that $\omega_h > 0$ for every $h = 1, \cdots, n$. Each agent i's preferences over goods and money are quasilinear: that is, the utility of agent i holding c units of money and the bundle x of goods can be expressed as $u_i(x, c) = V_i(x) + c$, where $V_i(x)$ is the reservation value, the quantity of money that agent i valuates the bundle x of goods. For each $i \in T$, the reservation value function $V_i : \mathbb{Z}^n \mapsto \mathbb{R}$ is assumed to be bounded from above. Furthermore, each agent i is assumed to have a sufficient amount m_i of money in the sense that $m_i \geq \sup_{x \in \mathbb{Z}^n} V_i(x) - V_i(\omega^i)$. Since V_i is bounded above, m_i is

finite. This market model will be represented by $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{Z}^n)$. Note that we do not require any monotonicity in this model. So this model covers the cases where some agents want to get rid of their commodities (namely economic bads) such as used cars or aging nuclear plants.

A family $(x^1, x^2, \cdots, x^m)$ of bundles $x^i \in \mathbb{Z}^n$ is called a (feasible) allocation if $\sum_{i \in T} x^i = \omega$. An allocation $(x^1, x^2, \cdots, x^m)$ is *(socially) efficient* if it is an optimal solution of the following problem:

$$\begin{array}{ll} \max & \sum_{i=1}^m V_i(y^i) \\ s.t. & \sum_{i=1}^m y^i = \omega \\ & y^i \in \mathbb{Z}^n,\ i = 1, 2, \cdots, m. \end{array} \tag{2.1}$$

A price vector $p \in \mathbb{R}^n$ indicates a price (units of money) for each good. Given a price vector $p \in \mathbb{R}^n$, the demand of goods by agent i is defined by

$$D_i(p) = \{x \mid (V_i(x) + p(\omega^i - x)) = \max\{V_i(y) + p(\omega^i - y) \mid p \cdot y \leq m_i + p \cdot \omega^i, \quad y \in \mathbb{Z}^n\}\}.$$

Note that $m_i \geq \sup_{x \in \mathbb{Z}^n} V_i(x) - V_i(\omega^i)$ for every $i \in T$. This implies that the budget constraint $p \cdot y \leq m_i + p \cdot \omega^i$ is redundant. Thus, the set $D_i(p)$ can be simplified as

$$D_i(p) = \{x \mid (V_i(x) - p \cdot x) = \max\{V_i(y) - p \cdot y \mid \quad y \in \mathbb{Z}^n\}\}.$$

A tuple $((x^1, x^2, \cdots, x^m); p)$ is a *Walrasian equilibrium* if p is a vector in $\mathbb{R}^n$; and if $x^i \in D_i(p)$ for every $i \in T$; and if $\sum_{i \in T} x^i = \omega$. The allocation $(x^1, x^2, \cdots, x^m)$ will be called *an equilibrium allocation.* Thus, in equilibrium, each agent gets his best bundle of goods under his budget constraint and moreover market is clear. The following simple lemma indicates that a free market mechanism will lead to a socially efficient allocation of resources.

Lemma 2.1 *Suppose that the allocation $(x^1, x^2, \cdots, x^m)$ is an equilibrium allocation. Then it must be socially efficient.*

The lemma shows that the equlibrium concept is indeed interesting and appealing. It is well known from Debreu (1959) that for market models with divisible goods there exists an equilibrium if every agent's utility function is concave and weakly increasing. Unfortunately, with indivisibilities, an equilibrium may not exist under similar conditions.

Recall that a function $f : \mathbb{Z}^n \mapsto \mathbb{R}$ is said to be *(discrete) concave* if, for any points x^1, x^2, $\cdots$, x^l in $\mathbb{Z}^n$ with any convex parameters λ_1, λ_2, $\cdots$, λ_l satisfying $\sum_{h=1}^{l} \lambda_h x^h \in \mathbb{Z}^n$, it holds

$$f(\sum_{h=1}^{l} \lambda_h x^h) \geq \sum_{h=1}^{l} \lambda_h f(x^h).$$

In particular, given an integral vector $\bar{y} \in \mathbb{Z}^n$, a function $f : \mathbb{Z}^n \mapsto \mathbb{R}$ is said to be *(discrete) concave with respect to* $\bar{y}$ if, for any points x^1, x^2, $\cdots$, x^l in $\mathbb{Z}^n$ with any convex parameters λ_1, λ_2, $\cdots$, λ_l satisfying $\bar{y} = \sum_{h=1}^{l} \lambda_h x^h$, it holds

$$f(\bar{y}) \geq \sum_{h=1}^{l} \lambda_h f(x^h).$$

Clearly, if f is a concave function, then f must be a concave function with respect to $\bar{y}$. The other way around is not true.

Here we slightly modify the example of Bevia, Quinzii and Silva (1999) to demonstrate that a simple modification of concavity to the indivisibility case is not sufficient to ensure the existence of an equilibrium. In an indivisible goods market, there are three agents 1, 2 and 3, and three indivisible goods. Agent 1 initially owns one unit of good 1 and 20 dollars, agent 2 owns one unit of good 2 and 20 dollars, and agent 3 owns one unit of good 3 and 20 dollars. Let $B^3 = \{x \in \mathbb{Z}^3 \mid 0 \leq x_i \leq 1, i = 1, 2, 3\}$. Their reservation value functions are given by $V_1(0,0,0) = 0$, $V_1(1,0,0) = 10$, $V_1(0,1,0) = 8$, $V_1(0,0,1) = 2$, $V_1(1,1,0) = 13$, $V_1(1,0,1) = 11$, $V_1(0,1,1) = 9$, $V_1(1,1,1) = 14$, $V_1(x) = \max\{V_1(y) \mid y \in B^3, y \leq x\}$ for $x \in \mathbb{Z}^3_+ \setminus B^3$, and $V_1(x) = -\infty$ if $x_i < 0$ for some i; $V_2(0,0,0) = 0$, $V_2(1,0,0) = 8$, $V_2(0,1,0) = 5$, $V_2(0,0,1) = 10$, $V_2(1,1,0) = 13$, $V_2(1,0,1) = 14$, $V_2(0,1,1) = 13$, $V_2(1,1,1) = 15$, $V_2(x) = \max\{V_2(y) \mid y \in B^3, y \leq x\}$ for $x \in \mathbb{Z}^3_+ \setminus B^3$, and $V_2(x) = -\infty$ if $x_i < 0$ for some i; $V_3(0,0,0) = 0$, $V_3(1,0,0) = 1$, $V_3(0,1,0) = 1$, $V_3(0,0,1) = 8$, $V_3(1,1,0) = 2$, $V_3(1,0,1) = 9$, $V_3(0,1,1) = 9$, $V_3(1,1,1) = 10$, $V_3(x) = \max\{V_3(y) \mid y \in B^3, y \leq x\}$ for $x \in \mathbb{Z}^3_+ \setminus B^3$, and $V_3(x) = -\infty$ if $x_i < 0$ for some i. Clearly, V_1, V_2, and V_3 are weakly increasing, (discrete) concave and bounded from above, and their marginal returns are decreasing. Although the reservation value functions seem to be extremely plausible, yet there is no equilibrium in this market. In fact, in this market, there is only one efficient allocation, namely, agent 1 gets $(0,1,0)$, agent 2 gets $(1,0,0)$, and agent 3 gets $(0,0,1)$.

Unfortunately, it can be shown that this allocation is not an equilibrium allocation. See Bevia et al. (1999) in detail. We will come back to this example later.

3 Equilibrium Existence Theorems

In this section we will establish existence results for the market model. Define the following potential market value function on $\mathbb{Z}^n$:

$$R(x) = \sup\{\sum_{i\in T} V_i(x^i) \mid \sum_{i\in T} x^i = x, x^i \in \mathbb{Z}^n\}.$$

$R(x)$ is the maximal market value that can be achieved by all the agents with the resource vector x. The function R is also called the *max-convolution function* generated by V_1, V_2, $\cdots$, V_m and the analysis based upon this function is called the *max-convolution approach*. We point out that this approach is somehow related to Negishi (1960) but differs considerably from his in that here $R(x)$ is the maximal market value under the resource x and is a function of the social endowment x, whereas Negishi defined a social welfare function as the weighted sum of utility functions where weights are variables and the social endowment is not treated as a variable but a given constant. His approach works for economies with divisible goods.

Our first result gives a necessary and sufficient condition for the existence of a Walrasian equilibrium. Recall that ω is the total initial endowment of indivisible commodities.

Lemma 3.1 *Given a market model $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{Z}^n)$, there exists a Walrasian equilibrium if and only if the following system of linear inequalities has a solution $p \in \mathbb{R}^n$*

$$p \cdot (x - \omega) \;\geq\; R(x) - R(\omega), \;\; \forall x \in \mathbb{Z}^n.$$

Proof: Suppose that $((x^{1*}, x^{2*}, \cdots, x^{m*}); p^*)$ is a Walrasian equilibrium. Then we have for all $i \in T$ and all $y \in \mathbb{Z}^n$ it holds

$$V_i(x^{i*}) - p^* \cdot x^{i*} \geq V_i(y) - p^* \cdot y. \tag{3.2}$$

It follows from Lemma 2.1 that $\sum_{i=1}^m V_i(x^{i*}) = R(\omega)$. For any $x \in \mathbb{Z}^n$, then there must exist $x^i \in \mathbb{Z}^n$ with $\sum_{i=1}^m x^i = x$ such that $\sum_{i=1}^m V_i(x^i) = R(x)$. It follows from (3.2) that

$$V_i(x^{i*}) - p^* \cdot x^{i*} \geq V_i(x^i) - p^* \cdot x^i, \;\; \forall i \in T.$$

Therefore we have $p^* \cdot (x - \omega) \geq R(x) - R(\omega),\ \forall x \in \mathbb{Z}^n$.

On the other hand, suppose that $p^* \in \mathbb{R}^n$ is a solution of the following system

$$p \cdot (x - \omega) \quad \geq \quad R(x) - R(\omega),\ \forall x \in \mathbb{Z}^n.$$

Let $(x^{1*}, x^{2*}, \cdots, x^{m*})$ be any allocation so that $\omega = \sum_{i=1}^m x^{i*}$ and $R(\omega) = \sum_{i=1}^m V_i(x^{i*})$ with $x^{i*} \in \mathbb{Z}^n$. Note that such allocation always exists by the definition of $R(x)$. We will show that $((x^{1*}, x^{2*}, \cdots, x^{m*}); p^*)$ is a Walrasian equilibrium. For any agent i and any $y \in \mathbb{Z}^n$, let $x = \sum_{l \neq i} x^{l*} + y$. By assumption we have

$$R(\omega) - p^* \cdot \omega \geq R(x) - p^* \cdot x.$$

By definition of $R(x)$, we have

$$R(x) \geq \sum_{l \neq i} V_l(x^{l*}) + V_i(y).$$

Therefore, it follows that

$$\begin{array}{rcl} \sum_{i=1}^m V_i(x^{i*}) - p^* \cdot \sum_{i=1}^m x^{i*} & = & R(\omega) - p^* \cdot \omega \\ & \geq & R(x) - p^* \cdot x \\ & \geq & \sum_{l \neq i} V_l(x^{l*}) + V_i(y) - p^* \cdot (\sum_{l \neq i} x^{l*} + y). \end{array}$$

The above implies that $V_i(x^{i*}) - p^* \cdot x^{i*} \geq V_i(y) - p^* \cdot y$.

Since i and y are taken arbitrarily, it is clear that $((x^{1*}, x^{2*}, \cdots, x^{m*}), p^*)$ is indeed a Walrasian equilibrium. □

In the above lemma, the equilibrium price of each good may be postive, zero, or even negative. The following lemma gives a rather weak condition to ensure that all goods have positive equilibrium prices.

Lemma 3.2 *Suppose that the market $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{Z}^n)$ has a Walrasian equilibrium. If $R(\omega + e(i)) > R(\omega)$ for all $i \in I_n$, then the equilibrium prices for all goods are positive.*

Now we are ready to present our main result which establishes a natural and intimate connection between Walrasian equilibrium and local concavity.

Theorem 3.3 *Given a market model $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{Z}^n)$, there exists a Walrasian equilibrium if and only if the market potential value function $R : \mathbb{Z}^n \mapsto \mathbb{R}$ is a discrete concave function with respect to ω.*

Proof: By Lemma 3.1 it is sufficient to show that the market potential value function R is a concave function with respect to ω if and only if the following system of linear inequalities has a solution $p \in \mathbb{R}^n$

$$p \cdot (x - \omega) \;\geq\; R(x) - R(\omega), \;\; \forall x \in \mathbb{Z}^n.$$

Suppose that p^* is a price vector satisfying the above inequalities. Let $x^1, x^2, \cdots, x^l \in \mathbb{Z}^n$ with convex parameters λ_1, λ_2, $\cdots$, λ_l such that $\omega = \sum_{h=1}^{l} \lambda_h x^h$. Since

$$R(\omega) - p^* \cdot \omega \geq R(x^h) - p^* \cdot x^h, \;\; h = 1, 2, \cdots, l,$$

and $\lambda_h \geq 0$ for $h = 1, 2, \cdots, l$, then we have

$$\lambda_h(R(\omega) - p^* \cdot \omega) \geq \lambda_h(R(x^h) - p^* \cdot x^h), \;\; h = 1, 2, \cdots, l.$$

Since $\sum_{h=1}^{l} \lambda_h = 1$ and $\omega = \sum_{h=1}^{l} \lambda_h x^h$, it follows that

$$R(\omega) \geq \sum_{h=1}^{l} \lambda_h R(x^h).$$

Thus, the potential market value function R is a concave function with respect to ω.

On the other hand, suppose that the potential market value function R is a concave function with respect to ω. Then, by definition, if ω is a convex combination of points x^1, x^2, $\cdots$, x^l in $\mathbb{Z}^n$ with convex parameters λ_1, λ_2, $\cdots$, λ_l, then we have

$$R(\omega) \geq \sum_{h=1}^{l} \lambda_h R(x^h). \tag{3.3}$$

Now let G be the graph of the function R, i.e., $G = \{(x, R(x)) \mid x \in \mathbb{Z}^n\}$. Let H be the convex hull of the set G, which is a closed convex set. Take an arbitrary point $(\omega, z) \in H$. Then there exist x^1, x^2, $\cdots$, x^l in $\mathbb{Z}^n$ with convex parameters λ_1, λ_2, $\cdots$, λ_l, such that $\omega = \sum_{h=1}^{l} \lambda_h x^h$ and $z = \sum_{h=1}^{l} \lambda_h R(x^h)$. It follows from (3.3) that

$$(\omega, R(\omega)) \geq (\omega, z).$$

This implies that $(\omega, R(\omega))$ is a boundary point of the set H. The well known separation theorem implies that there exists a nonzero vector $(-p, t) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$-p \cdot \omega + tR(\omega) \geq -p \cdot y + tz$$

for all $(y, z) \in H$. In particular, we have

$$-p \cdot \omega + tR(\omega) \geq -p \cdot x + tR(x), \tag{3.4}$$

for all $x \in \mathbb{Z}^n$.

Since $\omega_h > 0$ for all $h = 1, 2, \cdots, n$, and ω lies in the interior of $\mathbb{Z}^n$, it is easy to see that there does not exists any nonzero vector $p \in \mathbb{R}^n$ such that

$$-p \cdot \omega \geq -p \cdot x, \ \forall x \in \mathbb{Z}^n.$$

This means that $t \neq 0$. It follows from (3.3) and (3.4) that t can be made positive. Without loss of generality, we may assume $t = 1$. Now the system (3.4) implies

$$p \cdot (x - \omega) \geq R(x) - R(\omega), \ \forall x \in \mathbb{Z}^n.$$

The proof is complete. □

We now return to the previous non-existence example. For this example, the reservation value functions V_1, V_2, and V_3 are discrete concave functions on $\mathbb{Z}^3$. We have $R(1,1,1) = 24$, $R(1,0,1) = 20$, and $R((1,2,1)) = 29$. Because $R(1,1,1) = 24 < (R(1,1,1) + R(1,2,1))/2 = 24.5$, the function R is not concave with respect to $\omega = (1,1,1)$ and thus the market has no equilibrium.

Note that the conditions stated in both results above are imposed on the collective behaviors of all agents. In the next section we will provide sufficient conditions on the behaviors of each individual agent.

4 Max-convolution Concavity Preservable Functions

In this section we will identify agents' reservation value functions for the existence of Walrasian equilibrium and discuss the difference between the divisible goods market and

the indivisible goods market. For this purpose, we will introduce a class of new utility functions, called the class of max-convolution concavity preservable functions.

In the following, we assume that every function under consideration is bounded from above. Let f_1 and f_2 be functions mapping from $\mathbb{Z}^n$ to $\mathbb{R}$. Define $f_1 \oplus f_2 : \mathbb{Z}^n \mapsto \mathbb{R}$ by

$$f_1 \oplus f_2(x) = \sup\{f_1(x^1) + f_2(x^2) \mid x^1 + x^2 = x, x^1, x^2 \in \mathbb{Z}^n\}.$$

Definition 4.1 *A class $\mathcal{F} = \{f \mid f : \mathbb{Z}^n \mapsto \mathbb{R}\}$ of functions is said to be max-convolution concavity preservable if the following conditions are satisfied:*

(i) For every $f \in \mathcal{F}$, f is (discrete) concave;

(ii) For every f and g in $\mathcal{F}$, we also have $f \oplus g$ in $\mathcal{F}$.

A function f is said to be *max-convolution concavity preservable* if f is a member of some class of max-convolution concavity preservable functions. Similarly, functions $f_1, \cdots, f_m$ are said to be *max-convolution concavity preservable* if they belong to the same class of max-convolution concavity preservable functions. One can analogously define the above concepts for the continuous case.

It follows immediately from Theorem 3.3 and Definition 4.1 that given a market model $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{Z}^n)$, if reservation value functions V_i, $i \in T$, are max-convolution concavity preservable, then the market has a Walrasian equilibrium. In the following, we will offer several case studies.

When the commodities space is $\mathbb{R}^n$ (the divisible goods space), then we have the following simple lemma; see e.g., Rockafellar (1970).

Lemma 4.2 *If $V_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, 2$, are concave, then the function $V_1 \oplus V_2$ is also concave.*

Let $\mathcal{F} = \{f \mid f : \mathbb{R}^n \mapsto \mathbb{R}$ is concave $\}$. Then $\mathcal{F}$ is max-convolution concavity preservable. As a consequence, we have that every divisible goods market has a Walrasian equilibrium if reservation value functions $V_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i \in T$, are concave.

When the commodities space is $\mathbb{Z}^n$ (the indivisible goods space), things become much more complicated. The fundamental difference between the indivisible goods market and

the divisible goods market lies in the fact that for the divisible goods market all concave functions are max-convolution concavity preservable, whereas for the indivisible goods market, not all discrete concave functions are max-convolution concavity preservable. The non-existence example in Section 2 will help illustrate this point. Clearly, functions V_1, V_2 and V_3 are discrete concave functions on $\mathbb{Z}^3$. But, the max-convolution function R generated by V_1, V_2 and V_3 fails to be discrete concave as shown above.

Murota and Shioura (1999) have introduced a class of discrete concave functions which are max-convolution concavity preservable; see also Murota (1998, 2003). A function $f : \mathbb{Z}^n \mapsto \mathbb{R}$ is said to be *M$^\natural$-concave* if for every $x, y \in \mathbb{Z}^n$ and every $k \in \text{supp}^+(x-y)$ with $\text{supp}^+(x-y) \neq \emptyset$, it holds

$$\begin{aligned} f(x)+f(y) \;\leq\; & \max[f(x-e(k))+f(y+e(k)), \\ & \max_{l\in \text{supp}^-(x-y)}\{f(x-e(k)+e(l))+f(y+e(k)-e(l))\}] \end{aligned}$$

where $\text{supp}^+(x-y) = \{k \in I_n \mid x_k > y_k\}$ and $\text{supp}^-(x-y) = \{k \in I_n \mid x_k < y_k\}$.

The functions $f : \mathbb{Z}^n \mapsto \mathbb{R}$ given as $f(x) = a \cdot x + c$ with $a \in \mathbb{R}^n$, and as $f(x) = \sum_{i=1}^n g_i(x_i)$ where $g_i : \mathbb{Z} \mapsto \mathbb{R}$, $i \in I_n$, are discrete concave, are all simple examples of M$^\natural$-concave function. Note that an M$^\natural$-concave function is also discrete concave. The following result is due to Murota (2003).

Theorem 4.3 *If $V_i : \mathbb{Z}^n \mapsto \mathbb{R}$, $i = 1, 2$, are M$^\natural$-concave functions, then the function $V_1 \oplus V_2$ is also M$^\natural$-concave.*

Thus the class of M$^\natural$-concave functions is max-convolution concavity preservable. As a consequence of Theorems 3.3, 4.3, we have that every indivisible goods market $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{Z}^n)$, has a Walrasian equilibrium if reservation value functions $V_i : \mathbb{Z}^n \mapsto \mathbb{R}$, $i \in T$, are M$^\natural$-concave. Danilov et al. (2001), Fujishige and Yang (2002) derived a similar result using more sophisticated techniques.

Applying Lemma 3.1, Theorem 4.3 and a discrete separation theorem of Murota (2003), we have

Theorem 4.4 *Given a market model $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{Z}^n)$, there exists a Walrasian equilibrium with an integral equilibrium price vector $p^* \in \mathbb{Z}^n$, if $V_i : \mathbb{Z}^n \mapsto \mathbb{Z}$, $i = 1, 2, \cdots, m$, are M$^\natural$-concave functions.*

Proof: Since all reservation value functions V_i are $M^\natural$-concave and integer valued, the max-convolution function R generated by $V_1, \cdots, V_m$ is also $M^\natural$-concave and integer valued. It follows from Murota (2003) that there exists an integral vector $p^* \in \mathbb{Z}^n$ and $\beta^* \in \mathbb{Z}$ such that

$$\beta^* + p^* \cdot \omega = R(\omega),$$

and

$$\beta^* + p^* \cdot x \geq R(x), \ \forall\, x \in \mathbb{Z}^n.$$

These inequalities imply that

$$p^* \cdot (x - \omega) \geq R(x) - R(\omega), \ \forall\, x \in \mathbb{Z}^n.$$

By Lemma 3.1, we know that p^* is an equilibrium price vector. □

Note that in the theorem all functions V_i are integer valued and the equilibrium price vectore p^* is integral. So this model is more realistic in the sense that money can be also modeled as an indivisible good.

Finally, we discuss the well-known Kelso and Crawford's gross substitutes condition, which is widely used in the literature on equilibrium, matching and auction models.

Consider a market with m traders and n indivisible objects (or goods), denoted by $N = \{1, 2, \cdots, n\}$. Note that when there are identical objects, one may use different numbers to differentiate them. It is easy to show that at equilibrium, identical objects all have the same price. Each trader i has a reservation value function over the objects, denoted by $V_i : 2^N \to \mathbb{R}$, where 2^N is the collection of all subsets of N. In other words, V_i is a function mapping from the set $\{x \in \mathbb{Z}^n_+ \mid x \leq \sum_{i \in I_n} e(i)\}$ to $\mathbb{R}$. It is assumed that $V_i(\emptyset) = 0$ and V_i is weakly increasing. Given a price vector $p \in \mathbb{R}^n$, the demand set $D_i(p)$ of trader i is defined as

$$D_i(p) = \{S \mid V_i(S) - \textstyle\sum_{h \in S} p_h = \max\{V_i(T) - \sum_{h \in T} p_h \ \mid \ \ T \subseteq N\}\}.$$

For the existence of an equilibrium, Kelso and Crawford (1982) introduced the following condition with respect to the demand set $D_i(p)$, known as *gross substitutes* (GS).

(i) For any two price vectors p and q such that $p \leq q$, and any $A \in D_i(p)$, there exists $B \in D_i(q)$ such that $\{k \in A \mid p_k = q_k\} \subseteq B$.

Gul and Stacchetti (1999) have introduced the following two conditions, called *single improvement* (SI) and *no complementarities* (NC), and shown the equivalence between GS, SI, and NC:

(1) (SI): For any price vector p and $A \notin D_i(p)$ there exists $B \subseteq N$ such that $V_i(A) - \sum_{h \in A} p_h < V_i(B) - \sum_{h \in B} p_h$, $|A \setminus B| \leq 1$, and $|B \setminus A| \leq 1$.

(2) (NC): For any price vector p and any $A, B \in D_i(p)$ and $X \subseteq A$, there exists $Y \subseteq B$ such that $(A \setminus X) \cup Y \in D_i(p)$. □

Note that the GS, SI and NC properties are not conditions on the primitive characteristics of the economy (the reservation value functions) but conditions on the derived demand correspondences.

Fujishige and Yang (2003) have proved

Theorem 4.5 *A reservation value function $V : 2^N \mapsto \mathbb{R}$ satisfies the gross substitutes condition if and only if V is $M^\natural$-concave.*

So this result has identified the complete set of reservation value functions having the GS or SI or NC property. Three special classes of reservation value functions in this complete set were previously discovered by Kelso and Crawford (1982), Bevia et al. (1999), Gul and Stacchetti (1999). Danilov et al.(2003), Murota and Tamura (2003) have shown that all these results can be analogously extended from 2^N to $\mathbb{Z}^N$.

Note that when $M^\natural$-concave function is specified on a set function, it reads as follows: A set function $f : 2^N \to \mathbb{R}$ is an $M^\natural$-concave function if for each $S, T \subseteq N$ and $s \in S \setminus T$ with $S \setminus T \neq \emptyset$ the function f satisfies

$$f(S) + f(T) \leq \max[f(S - s) + f(T + s), \max_{t \in T \setminus S}\{f((S - s) + t) + f((T - t) + s)\}].$$

In the above formula, we read $S - s$ and $T + s$ as $S \setminus \{s\}$ and $T \cup \{s\}$, respectively.

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