Sectoral Labor Allocations, Economic Growth, and Income Inequality*

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Abstract
This paper develops a dynamic general equilibrium model with two sectors, multi-dimensional heterogeneity, and joint determination of growth and inequality. We use this framework to analyze the growth-inequality relationship both along the balanced growth path when fundamentals change and along the transitional trajectory as the economy converges to the balanced growth path. The main results are: (1) Fundamental changes affect income inequality by influencing the cross-sector labor allocation; (2) Physical and human capital heterogeneity play opposite roles in the way fundamental changes affect growth-inequality relationship; (3) Growth-inequality relationship along transitional trajectory may differ from the one along the balanced growth path, but relative importance of two types of heterogeneity matters in all cases. Therefore the ambiguity of empirical evidence on growth-inequality relationship should not be too surprising.

JEL classification: O41, D31.
Key words: Growth; Income inequality; Physical and human capital heterogeneity; Transitional dynamics.

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1 Introduction

The empirical evidence on the relationship between growth and income inequality has been inconclusive. Alesina and Rodrik (1994), Persson and Tabellini (1994), Perotti (1996), and others find a negative growth-inequality relationship. More recent studies, for example, Li and Zou (1998), Forbes (2000), and Lundberg and Squire (2003), however, obtain a positive, or at least more ambiguous result. Barro (2000) finds that the relationship between growth and inequality depends on the wealth level of each country: it tends to be negative for poorer countries, and be positive for richer countries.

Early theoretical models on this matter, for example, Alesina and Rodrik (1994), Persson and Tabellini (1994), take the initial inequality as given and focus on how inequality exerts a negative impact on rate of growth. Later papers start to build a framework where growth and inequality are both endogenous and subject to common forces, so they are jointly determined. Bertola (1993) shows how technological parameters jointly affect growth and inequality and how policies that increase growth affect inequality. But in his model inequality doesn’t respond to any policy changes because labor supply is assumed to be fixed. García-Penalosa and Turnovsky (2006) and Viane and Zilcha (2003) generate growth-inequality relationship with endogenous labor supply: the former works with physical capital heterogeneity and the latter assumes human capital heterogeneity instead. Most recently, Jin and Zeng (2006) and Jin (2006) combine two types of heterogeneity but human capital difference is assumed to be exogenous.

This paper attempts to achieve two goals. First it aims to develop a dynamic general equilibrium model with two sectors (goods producing sector and education sector), multi-dimensional heterogeneity (consumers dif-
fer in both physical and human capital holdings), and joint determination of growth and inequality. Second, we use this framework to analyze the growth-inequality relationship both along the balanced growth path when fundamentals change and along the transitional trajectory as the economy converges to the balanced growth path.

Our paper is closely related to Alesina and Rodrik (1994), García-Penalosa and Turnovsky (2006), Mulligan and Sala-i-Martin (1993), and Caselli and Ventura (2000). Alesina and Rodrik (1994) develop a one-sector AK model where agents are heterogeneous in their initial physical capital endowment. They show that income share is linear in the physical capital share with the coefficient being a function of the tax rate on the capital stock. If there is higher degree of inequality, median voter will have less capital compare to the average voter hence pick a higher tax rate, which hurts the economic growth. As a result, inequality has a negative impact on the rate of growth. Our paper doesn’t take the initial inequality as given, instead it allows both growth and inequality to be endogenous. Furthermore, it assumes that consumers also differ in human capital holdings. This additional dimension of heterogeneity becomes another source of income inequality. The individual income share thus becomes a convex combination of her physical and human capital shares.

García-Penalosa and Turnovsky (2006) build a one-sector AK model where growth and income inequality are jointly determined. Individual income share is also linear in the physical capital share, as in Alesina and Rodrik (1994), but the coefficient depends on the endogenous labor supply. Fundamental changes affect growth and inequality in the same direction, supporting a positive relationship between the two. Our paper differs from theirs in two important aspects. First, total labor supply in our model is fixed while
how the labor supply is allocated across the two sectors is endogenous. The individual income share is a convex combination of her physical and human capital shares, with coefficients varying with endogenous sectoral allocations of labor. Furthermore, later we show that the two types of heterogeneity play opposite roles in the variation of the income inequality. Second, AK models don’t have transitional dynamics. With a two-sector framework, sectoral technology difference (here it refers to different capital intensities) allows for transitional dynamics. This enables us to compare the growth-inequality relationship along the transitional trajectory with the one along the balanced growth path. In both cases, we show that the relative power of the two types of heterogeneity is crucial in the prediction of the growth-inequality relationship.

Mulligan and Sala-i-Martin (1993) present a two-sector model with homogenous agents. Their focus is to develop the “time-elimination method” to study transitional dynamics. Our paper adopts this method to convert a boundary condition problem to an initial value problem. Caselli and Ventura (2000)’s representative-agent theory of distribution also applies here. Despite the fact that agents are heterogenous, aggregate variables do not depend on the distribution.

We summarize our results here and compare with those of closely related papers. With two types of heterogeneity, the income share becomes a convex combination of physical and human capital shares, contrast to models with one-dimension heterogeneity where the income share is linear in either the physical or human capital share. Fundamental changes affect the income inequality through sectoral labor allocations with the aggregate labor supply being fixed. Physical and human capital heterogeneity play opposite roles in the growth-inequality relationship induced by fundamental changes.
When the physical capital heterogeneity dominates, changes in productivity and capital intensity cause a positive growth-inequality relationship while changes in the discount rate or the intertemporal elasticity of substitution cause a negative growth-inequality relationship along the balanced growth path. These results are consistent with what García-Peñalosa and Turnovsky obtain in their model. However, when the human capital heterogeneity dominates, all results are reversed. We also show that the growth-inequality relationship along the transitional trajectory may be different from the one along the balanced growth path. In particular, consider the case when physical capital heterogeneity dominates. If we consider the growth rate of full output, then a deviation of the capital stock ratio from its steady state level induces a positive growth-inequality relationship. But if growth is in terms of measured GDP, then the growth-inequality relationship depends on whether the physical-human capital stock ratio is higher or lower than its steady state value. This dependence of the growth measure is caused by the difference of the transitional dynamics for full output and measure GDP: full output monotonically decreases as the capital stock ratio increases while the dynamics of the measured GDP exhibit a U-shape. Once again when the human capital heterogeneity dominates, all results are reversed.

The rest of the paper is organized as the following. Section 2 lays out the model and derives first order conditions for the firms’ and consumers’ maximization problems. The balanced growth path is characterized in Section 3. We solve for the growth rate and income inequality in the steady state equilibrium. The different roles played by the two types of heterogeneity are emphasized. Then we investigate how growth is related to income inequality along the balanced growth path as fundamentals change. Section 4 further studies the transitional dynamics and explores the growth-inequality
relationship along the time path as the model economy converges to the balanced growth path. Last section summarizes major results. It also provides implications for empirical studies and future work.

2 The Model

2.1 Firms

There are two sectors in the economy, a final goods producing sector \((f)\) and an education sector \((e)\). Each sector adopts the following Cobb-Douglas technology:

\[
\begin{align*}
 f &= Z_f (K_f)^{\alpha_k} (L_f)^{1-\alpha_k}, \\
 e &= Z_e (K_e)^{\beta_k} (L_e)^{1-\beta_k},
\end{align*}
\]

where \(K_f (K_e)\) and \(L_f (L_e)\) represent physical capital stock and effective labor input used in sector \(f (e)\), respectively, and \(Z_f\) and \(Z_e\) are the technology parameters. Effective labor will be discussed later in the consumers’ problems.

Let \(p\) be the price of education service. Profit maximization in each sector gives the wage rate and rental rate (both in real terms),

\[
\begin{align*}
 w &= Z_f (1-\alpha_k) \left( K_f / L_f \right)^{\alpha_k} = p(1-\beta_k) Z_e (K_e / L_e)^{\beta_k}, \\
 r &= A\alpha_k \left( K_f / L_f \right)^{\alpha_k-1} = p\beta_k Z_e (K_e / L_e)^{\beta_k-1}.
\end{align*}
\]

2.2 Consumers

There are a continuum of consumers with unit mass. Consumers are indexed by \(i\). The initial physical and human capital stocks for consumer \(i\) are \(K_{i0}\) and \(H_{i0}\), respectively. Let \(k_i\) and \(h_i\) be consumer \(i\)’s capital shares,

\[
k_i \equiv \frac{K_i}{K}, \quad h_i \equiv \frac{H_i}{H},
\]
where $K$ and $H$ are corresponding aggregate variables. $k_i$ has a distribution function $G(k_i)$, with mean 1 and variance $\sigma_k^2$. Similarly, $h_i$ has a distribution function $Q(h_i)$, with mean 1 and variance $\sigma_h^2$. Time endowment is normalized to be unity.

Let $\beta$ be the time discount rate. Consumer $i$ solves the following problem:

$$\max_{c_i, K_i, H_i, e_i} \int_0^\infty \frac{c_i^\gamma}{\gamma} e^{-\beta t} dt, \quad \gamma < 0$$

subject to

$$\dot{K}_i = rK_i + w (1 \cdot H_i) - c_i - p e_i - \delta_k K_i, \quad (5)$$

$$\dot{H}_i = e_i^\theta H_i^{1-\theta} - \delta_h H_i, \quad (6)$$

$$K_{i0}, H_{i0} > 0, \quad \text{given.}$$

Equation (5) is the budget constraint. In each period, consumer $i$ rents physical capital $K_i$ to firms and supplies one unit of labor inelastically, providing $(1 \cdot H_i)$ units of human capital augmented labor, or effective labor. Besides consumption $c_i$, the consumer also purchases $e_i$ units of education services and invests in the physical capital stock $K_i$. Equation (6) describes the human capital evolution process. The gross human capital investment, $e_i^\theta H_i^{1-\theta}$, combines both the education services received this period and the existing human capital stock to capture the fact that agents with higher human capital stock are generally more efficient learners. Also note that our model nests Mulligan and Sala-i-Martin (1993)’s no-externality model as a special case: when $\theta = 1$, our model reduces to a competitive equilibrium version of their model without externalities. The physical and human capital are assumed to depreciate at a rate $\delta_k$ and $\delta_h$, respectively.

Let $\lambda_i$ and $\mu_i$ be the Lagrangian multipliers associated with constraints
(5) and (6), respectively. The current-value Hamiltonian function is
\[ \mathcal{H} = \frac{c_i^2}{\gamma} + \lambda_i (rK_i + wH_i - c_i - pe_i - \delta K_i) + \mu_i (e_i^\theta H_i^{1-\theta} - \delta_h H_i). \]

First order conditions are given by
\[ c_i : c_i^{-1} = \lambda_i, \quad (7) \]
\[ e_i : \lambda_i \rho = \mu_i \theta e_i^{\theta - 1} H_i^{1-\theta}, \quad (8) \]
\[ K_i : \dot{\lambda}_i - \beta \lambda_i = -\lambda_i (r - \delta), \quad (9) \]
\[ H_i : \dot{\mu}_i - \beta \mu_i = -\left[ \lambda_i w + \mu_i (1 - \theta) e_i^\theta H_i^{1-\theta} - \mu_i \delta_h \right], \quad (10) \]

with the transversality conditions
\[ \lim_{t \to \infty} \lambda_i K_i e^{-\beta t} = 0, \quad \text{and} \quad \lim_{t \to \infty} \mu_i H_i e^{-\beta t} = 0. \quad (11) \]

### 2.3 Markets Clear

Market clearing requires demand for factors of inputs be equal to their supplies

\[ \Sigma K_i = K = K^f + K^e, \]
\[ \Sigma H_i = H = L^f + L^e, \]

and markets of two outputs in the two sectors also clear:

\[ e = \Sigma e_i, \quad (12) \]
\[ f = \Sigma \left( c_i + \dot{K}_i - (1 - \delta) K_i \right). \quad (13) \]

### 2.4 Equilibrium Conditions

First, note that equation (9) implies that the growth rate of \( \lambda \) is the same across consumers:
\[ \frac{\dot{\lambda}_i}{\lambda_i} = \beta - (r - \delta) = \frac{\dot{\lambda}}{\lambda}. \quad (14) \]
Taking time derivative of equation (7) gives

\[
\frac{\dot{c}_i}{c_i} = \frac{\dot{c}}{c} = \frac{1}{\gamma - 1} = \frac{r - \delta_k - \beta}{1 - \gamma}.
\]  

(15)

Thus the growth rate for consumption is independent of \(i\).

Define \(z \equiv \frac{K}{H}, a \equiv \frac{\kappa}{K}, b \equiv \frac{e}{H}, \) and \(z_i \equiv \frac{K_i}{H_i}, a_i \equiv \frac{\kappa_i}{K_i}, b_i \equiv \frac{e_i}{H_i}\) for all \(i\). Take time derivative of equation (8) and use (10), we have

\[
\frac{\dot{b}_i}{b_i} = \frac{1}{(\theta - 1)} \left[ -(r - \delta_k) + \frac{w}{p} \theta b_i^{\theta - 1} + (1 - \theta) b_i^\theta - \delta_k + \frac{\dot{\theta}}{\theta} \right].
\]  

(16)

Rewrite equations (5) and (6) as

\[
\frac{\dot{K}_i}{K_i} = r + \frac{w}{z_i} - a_i - p \frac{b_i}{z_i} - \delta_k, \quad (17)
\]

\[
\frac{\dot{H}_i}{H_i} = b_i^\theta - \delta_h. \quad (18)
\]

Aggregating the above yields

\[
\frac{\ddot{K}}{K} = r + \frac{w}{z} - a - p \frac{b}{z} - \delta_k, \quad (19)
\]

\[
\frac{\ddot{H}}{H} = \sum b_i^\theta \frac{H_i}{H} - \delta_h. \quad (20)
\]

Let \(v\) and \(u\) be the fractions of total physical capital and labor allocated to the goods producing sector. That is, \(v \equiv \frac{K^f}{K}\) and \(u \equiv \frac{L^f}{H}\). Using the wage and rental rates specified in equations (3) and (4) we derive the relationship between the allocations of capital and labor across sectors:

\[
v = \frac{\alpha_k (1 - \beta_k) u}{\alpha_k (1 - \beta_k) u + \beta_k (1 - \alpha_k) (1 - u)}, \quad (21)
\]

and

\[
\frac{\ddot{v}}{v} = \frac{\ddot{u}}{u} \frac{1 - v}{1 - u}. \quad (22)
\]
Take time derivative of the wage equation (3) and the market clearing condition for education sector (12), we have (after some algebra)

\[
\frac{\dot{u}}{u} = \frac{(\theta \beta_k - \alpha_k) \frac{\dot{K}}{K} + \left[ \theta (1 - \beta_k) - (1 - \alpha_k) \right] \frac{\dot{H}}{H} + \theta \delta_h + (r - \delta_k) - \frac{w \sum b_i H_i}{p \theta bH}}{\frac{(\theta \beta_k - \alpha_k) v + \theta (1 - \beta_k) - (1 - \alpha_k) w}{1 - u}},
\]

and

\[
\frac{\dot{p}}{p} = \left[ \beta_k \frac{\dot{K}}{K} + (1 - \beta_k) \frac{\dot{H}}{H} - \frac{\dot{u} \beta_k v + (1 - \beta_k) u}{u} \right] (\theta - 1) + \theta \delta_h + (r - \delta_k) - \frac{w \sum b_i H_i}{p \theta bH}.
\]

Using definitions of \( u, v, \) and \( z \) we can rewrite the wage and rental rates as

\[
r = A \alpha_k \left( \frac{v}{u} \right)^{\alpha_k - 1}, \tag{25}
\]

\[
w = Z_f (1 - \alpha_k) \left( \frac{v}{u} \right)^{\alpha_k}. \tag{26}
\]

Finally the price of education services and \( \frac{f}{\theta} \) are given by

\[
p = \frac{Z_f (1 - \alpha_k) \left( \frac{v}{u} \right)^{\alpha_k}}{Z^e (1 - \beta_k) \left( \frac{1 - w}{1 - u} \right)^{\beta_k}}, \tag{27}
\]

\[
b = Z^e \left( \frac{1 - v}{1 - u} \right)^{\beta_k} (1 - u). \tag{28}
\]

In summary, the equilibrium conditions include equations (15) – (24).

### 3 The Balanced Growth Path (BGP)

Along the balanced growth path, all variables grow at constant (possibly different) rates. From equation (15), constant \( \frac{\dot{z}}{\dot{c}} \) implies that \( r \) is constant, which then implies that \( \frac{v}{u} \) and \( w \) are also constant. Equation (21) shows \( \frac{1 - w}{1 - u} = \frac{(1 - \alpha_k) \beta_k v}{\alpha_k (1 - \beta_k) u} \), thus \( p \) is constant. From equation (16), constant \( \frac{b_i}{b_i} \) implies that \( b_i \) is constant (\( \dot{b}_i = 0 \)), hence \( b_i \) is independent of \( i \), or \( b_i = b \) for all \( i \).
The constancy of $b$ then leads to constancy of $u$ and $v$. Finally $z$ is constant. Equation (19) gives that $a$ is constant.

Now let’s consider physical and human capital shares. We present the following lemma.

**Lemma 1** Along BGP, individual’s physical and human capital shares remain constant: $\dot{k}_i = \dot{h}_i = 0$, for all $i$.

**Proof.** Since $b_i = b$ for all $i$, then $\frac{\dot{b}_i}{b_i} = \frac{\dot{H}_i}{H_i} - \frac{\dot{h}_i}{h_i} = 0$. The behavior of $k$ is less obvious. We write $\dot{k}_i$ as

$$\dot{k}_i = \left( \frac{\dot{K}_i}{K_i} - \frac{\dot{\bar{K}}}{\bar{K}} \right) k_i = \frac{w - pb}{z} h_i - \frac{c_i}{c} a - \left( \frac{w - pb}{z} - a \right) k_i.$$  

If (11) holds for all individuals, it also holds at the aggregate level, i.e.,

$$\lim_{t \to \infty} \lambda Ke^{-\beta t} = 0, \quad \text{and} \quad \lim_{t \to \infty} \mu He^{-\beta t} = 0. \quad (29)$$

The aggregate transversality conditions (29) hold if and only if $\dot{\bar{x}} + \frac{\dot{\bar{K}}}{\bar{K}} + \frac{\dot{a}(e^{-\beta t})}{e^{-\beta t}} < 0$. Using (14) and (19) the above condition reduces to

$$\frac{w - pb}{z} - a < 0.$$  

And furthermore, $\frac{c_i}{c}$ is constant ($c_i$ grows at the same rate as $c$). So $\dot{k}_i$ is linear in $k_i$ with the coefficient of $k_i$ being positive. To be consistent with long run equilibrium, we must have $\dot{k}_i = 0$. ■

The balanced growth path can be determined as follows. The steady state values of $a$, $u$, $z$, and $b$ are given by

$$\frac{\dot{a}}{a} = 0, \quad \frac{\dot{u}}{u} = 0, \quad \frac{\dot{z}}{z} = 0, \quad \frac{\dot{b}}{b} = 0, \quad (30)$$

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with $\hat{\epsilon}$, $\hat{K}$, $\hat{H}$, $\hat{v}$, $\hat{p}$, and $\hat{b}$ being specified by equations (15), (19), (20), (22), (23), (24), and (16). Equation (21) yields $v$. Note that the steady state value for $a_i$ and $z_i$ depend on the actual value of $k_i$ and $h_i$ which are determined by their distributions.

### 3.1 Income Inequality along BGP

We have shown that there exists a common economy-wide growth rate, which is independent of the initial distribution of physical and human capital holdings. Next we characterize income inequality along the balanced growth path.

Consumer $i$’s income $Y_i$ consists of capital rental income and wage income,

$$Y_i = rK_i + wH_i.$$ 

The relative income, or the income share, denoted as $y_i$, is defined to be the ratio of individual income $Y_i$ to the aggregate income $Y$:

$$y_i \equiv \frac{Y_i}{Y} = \frac{rK_i + wH_i}{f + pe}.$$ 

This share is our measure of the income inequality. It has a mean of 1 and variance $\sigma^2_y$. We can then rewrite $y_i$ as

$$y_i = \rho k_i + (1 - \rho) h_i,$$

where $\rho = \frac{rK}{f + pe}$, the share of physical capital income in the total income. The coefficient $\rho$ can be further expressed as

$$\rho = \frac{\alpha_k (1 - \beta_k) u + \beta_k (1 - \alpha_k) (1 - u)}{(1 - \beta_k) u + (1 - \alpha_k) (1 - u)}.$$ 

Interestingly, $\rho$ depends solely on the cross-sector labor allocations. Obviously

$$0 < \rho < 1,$$

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so \( y_i \) is increasing in both \( k_i \) and \( h_i \), and \( y_i - 1 = \rho (k_i - 1) + (1 - \rho) (h_i - 1) \).

Thus

\[
\sigma_y^2 = \rho^2 \sigma_k^2 + (1 - \rho)^2 \sigma_h^2 + 2\rho (1 - \rho) \sigma_{k,h}
\]

where \( \sigma_{k,h} \) is the covariance between \( k \) and \( h \). For simplicity we assume that \( \sigma_{k,h} = 0 \). Therefore

\[
\sigma_y^2 = \rho^2 \sigma_k^2 + (1 - \rho)^2 \sigma_h^2.
\]

We summarize the above results in the following proposition.

**Proposition 1**

(i) \( y_i \) is a convex combination of \( k_i \) and \( h_i \).

(ii) Fundamental changes affect \( \sigma_y^2 \) through their effects on the sectoral labor allocation. Changes in \( \alpha_k \) and \( \beta_k \) have direct distributional effects.

(iii). The physical capital income share, \( \rho \), is positively related to the labor allocated to the more capital intensive sector. That is, \( \frac{d\rho}{du} > 0 \) when \( \alpha_k > \beta_k \), and \( \frac{d\rho}{d(1-u)} > 0 \) when \( \alpha_k < \beta_k \).

(iv). The two types of capital heterogeneity play opposite roles in the effect of \( \rho \) on income inequality, that is, 

\[
\frac{\partial (\partial \sigma_y^2 / \partial \rho)}{\partial \sigma_k^2} \left[ \frac{\partial (\partial \sigma_y^2 / \partial \rho)}{\partial \sigma_h^2} \right] < 0.
\]

**Proof.** (i) and (ii) are directly implied by equations (31) and (32). (iii), we calculate

\[
\frac{d\rho}{du} = \frac{(\alpha_k - \beta_k) (1 - \alpha_k) (1 - \beta_k)}{[(1 - \beta_k) u + (1 - \alpha_k) (1 - u)]^2}.
\]

Clearly, if \( \alpha_k > \beta_k \), then \( \frac{d\rho}{du} > 0 \), and if \( \alpha_k < \beta_k \), then \( \frac{d\rho}{du} < 0 \). (iv), from (33) we have \( \frac{\partial \sigma_y^2 / \partial \rho}{\partial \sigma_k^2} = 2\rho \sigma_k^2 - 2 (1 - \rho) \sigma_h^2 \). Thus \( \frac{\partial (\partial \sigma_y^2 / \partial \rho)}{\partial \sigma_k^2} = 2\rho \) and \( \frac{\partial (\partial \sigma_y^2 / \partial \rho)}{\partial \sigma_h^2} = -2 (1 - \rho) \). 

Note that with the same technology in the two sectors, \( \rho \) is simply the common capital intensity, \( \rho = \alpha_k = \beta_k \), and hence invariant to fundamental changes other than \( \alpha_k \) or \( \beta_k \).
3.2 Growth-Inequality Relationship along BGP

Proposition 1 implies that how changes in $\rho$ affect the income inequality depends on the relative size of $\sigma_k^2$ and $\sigma_h^2$. If $\sigma_k^2 > \frac{1-\rho}{\rho} \sigma_h^2$ and hence $\frac{d\sigma_k^2}{d\rho} > 0$, we say the physical capital heterogeneity dominates. If $\sigma_k^2 < \frac{1-\rho}{\rho} \sigma_h^2$ and hence $\frac{d\sigma_k^2}{d\rho} < 0$, we say the human capital heterogeneity dominates. As a result, the growth-inequality relationship along the balanced growth path when fundamentals change also depends on which type of heterogeneity dominates. We present our results in the following proposition.

**Proposition 2** Along BGP,

(i). An increase in $Z_f$ or $Z_e$ raises the growth rate, inducing a positive (negative) growth-inequality relationship when physical (human) capital heterogeneity dominates;

(ii). A decrease in $\beta$ or an increase in $\gamma$ raise the growth rate, inducing a negative (positive) growth-inequality relationship when physical (human) capital heterogeneity dominates;

(iii). An increase in capital intensity $\alpha_k$ or $\beta_k$ has similar effect as in (i).

**Proof.** See the appendix. ■

Next we demonstrate the growth-inequality relationship numerically and graphically. Our numerical simulations use the following baseline values for the parameters, $\alpha_k = 0.4$, $\beta_k = 0.2$, $\delta_k = \delta_h = 0.05$, $\theta = 0.75$, and $Z_f = 1$. The discount rate $\beta$ equals 0.04. The level of technology in the education sector, $Z_e$, is chosen so as to obtain an annual steady-state growth rate equal to 0.015. The inverse of the intertemporal elasticity of substitution, $1 - \gamma$, is generally estimated to be larger than 1. We set $\gamma$ to be $-1$. The absolute size of $\sigma_k^2$ and $\sigma_h^2$ is irrelevant so we set $(\sigma_k^2, \sigma_h^2) = (1, 0)$ for the
case where physical capital heterogeneity dominates and \((\sigma_k^2, \sigma_h^2) = (0, 1)\) for the case where human capital heterogeneity dominates. With these baseline parameters, the steady state values for \(\psi, u, \rho, \) and \(\sigma_y^2\) are given in the second row of Table 1.

Then we change the parameters in a way such that growth rate always increases. The effects of these fundamental changes on the growth rate and income inequality are presented in Table 1. Intuition.

<table>
<thead>
<tr>
<th>Table 1: Growth-inequality relationship along BGP</th>
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<tbody>
<tr>
<td>(\psi)</td>
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<tr>
<td>baseline</td>
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<tr>
<td>(Z^I\uparrow)</td>
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<tr>
<td>(Z^e\uparrow)</td>
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<td>(\beta\downarrow)</td>
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<td>(\alpha_k\uparrow)</td>
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<td>(\beta_k\uparrow)</td>
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For further demonstration, we plot in Figure 1 and 2 the growth-inequality relationship along the balanced growth path. The horizontal axis measures the growth rate and the vertical axis measures inequality (income variance). For comparison purpose, we change parameters such that they all cause the growth rate to vary between 0.5% to 5%. Figure 1 corresponds to the case when the physical capital heterogeneity dominates. The two upward-sloping curves in panel (1) and (2) are generated by productivity changes, showing a slight positive growth-inequality relationship. Panels (3) and (4) clearly show a negative growth-inequality relationship induced by changes in \(\beta\) and \(\gamma\). Since \(\alpha_k\) and \(\beta_k\) are restricted to be between 0 and 1 and \(\alpha_k > \beta_k\), we
consider the case when $\alpha_k$ changes from the baseline value of $\beta_k$ (0.2) to 0.9, and $\beta_k$ changes from 0.1 to the baseline value of $\alpha_k$ (0.4). (Of course the induced growth rate changes are not limited between 0.5% and 5%.) Note that changes in capital intensities in two sector both generate positive growth-inequality relationship when physical capital heterogeneity dominates.

When human capital heterogeneity dominates (see Figure 2), we reverse all results in Figure 1. Changes in productivity in the two sectors now induce a slight negative growth-inequality relationship while changes in $\beta$ and $\gamma$ induce a significant positive relationship, and capital intensities generate negative growth-inequality relationship. This reverseness reflects the fact that when one moves from the dominance of $\sigma_k^2$ to the dominance of $\sigma_h^2$, fundamental changes affect income inequality oppositely whereas they affect the growth rate in the same way.

In both figures it is obvious that the slopes in panels (3) and (4) are much larger than the slopes in Panels (1) and (2). We characterize this in the following proposition.

**Proposition 3** For a given change in growth rate, a productivity increase has much smaller effects on inequality than changes in the rate of time preferences and the intertemporal elasticity of substitution with baseline parameter values.

**Proof.** See the appendix. ■

In summary, when physical capital heterogeneity dominates, our results on growth-inequality relationship along the balanced growth path are consistent with what is predicted in García-Penalosa and Turnovsky’s one-sector single-heterogeneity model. When human capital heterogeneity dominates, however, all results are reversed.
4 Transitional Dynamics and Growth-Inequality Relationship

In this section we present transitional dynamics of the model off its balanced growth path using time-elimination method described in Mulligan and Sala-i-Martin (1993).¹

We begin our analysis again by presenting properties of physical and human capital shares off the balanced growth path. In particular, we show that the steady state distributions of physical and human capital shares will be maintained along transitional trajectories. Their distributions are time-invariant.

Lemma 2 (i) \( \dot{k}_i = 0 \), for all \( i \) and at all time periods. (ii) During periods when \( \frac{w}{p} > b_i \), \( b_i = b \) for all \( i \), that is, \( \dot{h}_i = 0 \).

Proof. (i) At all time periods we have \( \frac{a_i}{a} = \frac{c_i}{c} \) and \( \frac{\dot{k}_i}{k_i} = -\left( \frac{\dot{a}_i}{a_i} - \frac{\dot{a}}{a} \right) \), where \( \frac{c_i}{c} \) is always constant since \( \frac{\dot{c}_i}{c_i} = \frac{\dot{c}}{c} \). Suppose \( k_i \) is higher than the initial value \( k_{i0} \), then \( \frac{a_i}{a} \) must be lower, or \( \frac{\dot{a}_i}{a_i} < \frac{\dot{a}}{a} \) or \( \frac{\dot{k}_i}{k_i} > 0 \). That is, \( k_i \) will increase continuously. This contradicts the fact that \( k_i \) is constant when the economy eventually converges to the balanced growth path. On the contrary, if \( k_i \) is lower than the initial value, then \( k_i \) will decrease continuously. Contradiction again. Thus \( k \) remains at the initial value.

(ii). Taking time derivative of the transversality condition of \( H_i \) yields \( \theta b_i^{\theta-1} \left( -\frac{w}{p} + b_i \right) < 0 \). That is, asymptotically \( \frac{w}{p} > b_i \), for all \( i \). Suppose \( \frac{w}{p} \) starts to be greater than \( b_i \) at period \( T \). The growth rate of \( b_i \), given by

¹Our emodel reduces to theirs when \( \theta = 1 \). Then if \( \alpha_k = \beta_k \), there is no transitional dynamics. When \( 0 < \theta < 1 \), however, transitional dynamics even exist when the two sectors have the same technology, \( \alpha_k = \beta_k \). For extensive discussion of conditions under which transitional dynamics exist, see Mulligan and Sala-i-Martin (1993).
equation (16), has the same form for all consumers. And

\[ \frac{\partial b}{\partial b} = \theta b^{\theta - 2} \left( \frac{w}{p} - b \right) > 0 \]

at period \( T \) and after

That is, if \( b_j \geq b_k \) at period \( T \), then \( \frac{b_j}{b_j} \geq \frac{b_k}{b_k} \), implying that \( b_j \) and \( b_k \) will never converge to the same value. Consequently we must have \( b_j = b_k \) at period \( T \). This holds no matter \( \frac{b_j}{b_k} \) is positive or negative. Thus after period \( T \), \( \frac{h_i}{h_i} = \frac{H_i}{H_i} = 0. \]

Lemma 2 implies that as long as an economy doesn’t deviate too far away from its balanced growth path, the human capital share distribution remains fixed along the transitional trajectory.

4.1 Apply Time-Elimination Method

In this section we use an example of two consumers. The analysis here applies for a continuum of consumers. State variables are the physical-human capital stock ratio \( z \), consumer 1’s capital shares \( k_1 \) and \( h_1 \). Other variables are functions of state variables, such as \( k_2 = 1 - k_1, h_2 = 1 - h_1, z_1 = \frac{k_1}{h_1} z, \) and \( z_2 = \frac{k_2}{h_2} z \).

When we solve for the balanced growth path, we transform a system of differential equations for \( c, u, K, H, \) and \( e \) for which there exists a constant growth path into a system of differential equations for \( a, u, z, \) and \( b \) for which there exists a stationary point. The stationary point \( (a^*, u^*, z^*, b^*) \) satisfies the transversality conditions, so does the stable manifold which is the locus of points in the \( [a, u, z, b] \) space which, when allowed to evolve according to (30), asymptotically approach the stationary point. Therefore the stable manifold describes optimal solutions. Alternatively we can represent the stable manifold by policy functions \( a(z), u(z), \) and \( b(z). \) Apply the chain
rule of calculus to construct a system of differential equations for policy functions rather than time paths as the following: 

\[
\begin{align*}
\dot{a} &= \frac{\dot{a}(t)}{\dot{z}(t)} = \frac{\left(\frac{c}{c} - \frac{\dot{K}}{K}\right)a}{\left(\frac{\dot{K}}{K} - \frac{\dot{H}}{H}\right)z}, \\
\dot{u} &= \frac{\dot{u}(t)}{\dot{z}(t)} = \frac{\left(\frac{\dot{u}}{u}\right)u}{\left(\frac{\dot{K}}{K} - \frac{\dot{H}}{H}\right)z}, \\
\dot{b} &= \frac{\dot{b}(t)}{\dot{z}(t)} = \frac{\frac{\dot{b}}{b}b}{\left(\frac{\dot{K}}{K} - \frac{\dot{H}}{H}\right)z},
\end{align*}
\]

(34) (35) (36)

Note that Lemma 2 also implies \( \frac{\dot{a}_i}{a_i} = \frac{\dot{a}}{a} \), for all \( i \). As a result, the steady state of \( a_i \) is given by:

\[
a_i = \left(\frac{w}{z} - \frac{pb}{z}\right)\left(\frac{h_i}{k_i} - 1\right) + a.
\]

The dynamics of \( a_i \) thus only depend on the aggregate state \( z \), just like other aggregate variables. The policy function \( a_1(z) \) is given by:

\[
\begin{align*}
\dot{a}_1 &= \frac{\dot{a}_1(t)}{\dot{z}(t)} = \frac{\left(\frac{c_1}{c_1} - \frac{\dot{K}_1}{K_1}\right)a_1}{\left(\frac{\dot{K}}{K} - \frac{\dot{H}}{H}\right)z} = \frac{\left(\frac{c}{c} - \frac{\dot{K}}{K}\right)a_1}{\left(\frac{\dot{K}}{K} - \frac{\dot{H}}{H}\right)z},
\end{align*}
\]

(37)

and \( a_2 = \frac{a-a_1k_1}{k_2} \).

4.2 Growth-Inequality Relationship along Transitional Trajectory

There are only two types of agents, then we cannot directly use the income share variance formula (33) derived for a continuum of agents. Instead, we

\[\text{We need to specify the slopes of the policy function at the steady state because the system (34) \rightarrow (37) cannot be applied directly. One can linearize (30) around the steady state and study the eigenvectors of the Jacobian matrix of (30) evaluated at the steady state. The stable manifold theorem guarantees that the stable eigenvector (the one that is associated with the stable eigenvalue) is exactly equal to the slope of the stable manifold of the (30) at the steady state.}\]
calculate the income inequality according to

\[ \sigma_y^2 = \frac{1}{2} \left[ (y_1 - \bar{y})^2 + (y_2 - \bar{y})^2 \right], \]

where the mean of the income share \( \bar{y} = \frac{1}{2} \). We consider two cases here. (1). \( k_1 = 0.25 \) and \( h_1 = 0.5 \). Thus \( \sigma_k^2 = \frac{1}{4} \) and \( \sigma_h^2 = 0 \), corresponding to the situation where physical capital heterogeneity dominates. (2). \( k_1 = 0.5 \) and \( h_1 = 0.25 \). Thus \( \sigma_k^2 = 0 \) and \( \sigma_h^2 = \frac{1}{4} \), corresponding to the situation where human capital heterogeneity dominates. The case where \( k_1 = h_1 \) is excluded here because \( \sigma_y^2 \) will be equal to \( \sigma_k^2 \) or \( \sigma_h^2 \) hence the income inequality will be a constant and won’t be affected by fundamental changes.

Transitional dynamics of some important variables are depicted in Figure 3. The horizontal axes measures the distance between \( z \) and its steady state value \( z^* \). Panels (1) and (2) show that along a transition where \( z \) is rising, the growth rate of physical capital will be falling and the growth rate of human capital is rising. A low value of \( z \) is raised through an increase in the allocation \( u \) to the final goods production. Thus \( u \) is high for low values of \( z \) (see panel (3)). Since \( \rho \) is positively related to \( u \) when \( \alpha_k > \beta_k \), it is also downward sloping as in panel (4). Growth rates of various outputs are reported in panels (5) – (8). The growth rate of goods output is downward sloping when it goes through the steady state, and it becomes an increasing function of \( z \) for larger values of \( z \). The growth rate of the production in the education sector is falling monotonically. The full output is given by

\[ Y^{full} = f + pe, \]

implying that

\[ \frac{\dot{Y}^{full}}{Y^{full}} = \frac{\dot{f}}{f} - \frac{1 - \alpha_k}{(1 - \beta_k) u + (1 - \alpha_k) (1 - u) u} \]

This growth rate is depicted in panel (7), unambiguously downward sloping for all values of \( z \). Following Mulligan and Sala-i-Martin, measured GDP
includes about 25% of the education sector, then the growth rate of measured GDP behaves somewhere between the growth rates presented in panels (5) and (7). It displays a U-shape with a minimum to the left of the steady state (panel (8)).

All transitional dynamics reported here are consistent with Mulligan and Sala-i-Martin’s general model with different technologies in the two sectors. Next we will see that the monotonic growth rate of full output and the U-shape growth rate of measured GDP have different implications in the growth-inequality relationship. We first consider the full output. Figure 4 plots the time paths for the growth rate of full output and income inequality. The horizontal axis measures years. Panel (1) reports the case when the capital stock ratio \( z \) is higher than its steady state value and physical capital heterogeneity dominates. From the transitional dynamics in panel (7) of figure 3, the growth rate of full output is increasing as \( z \) decreases. When the physical capital heterogeneity dominates, the variance of the income share \( \sigma^2_y \) moves in the same direction as \( \rho \) which is increasing over time. Thus panel (1) supports a positive growth-inequality relationship. If \( z \) is lower than its steady state value, the growth rate of full output and income share variance both decrease over time, again supporting a positive-inequality relationship. Panels (3) and (4) show the cases when the human capital heterogeneity dominates. The time paths of the growth rate of full output are the same as in panels (1) and (2), but results on income inequality are reversed. When human capital heterogeneity dominates, \( \sigma^2_y \) moves in the same direction as the coefficient of \( \sigma^2_n \) in the formula of income share variance. Since \( \rho \) is rising when \( z > z^* \) and decreasing when \( z < z^* \), \( \sigma^2_y \) will decrease when \( z > z^* \) and decrease when \( z < z^* \). As a result, when \( z \) deviates from \( z^* \), no matter \( z > z^* \) or \( z < z^* \), physical capital heterogeneity tends to generate positive growth-
inequality relationship along transition while human capital heterogeneity tends to generate negative growth-inequality relationship.

When we consider the growth rate of measured GDP, however, growth-inequality relationship can be different along a transition where $z$ is rising or falling. Figure 5 depicts the time paths for the growth rate of measured GDP and income inequality. All panels correspond to the same cases as in Figure 4. Compare to Figure 4, the time paths for the variance of income share are exactly the same. Furthermore, the time paths for the growth rate of output do not depend on which type of heterogeneity dominates. The major difference lies in the time paths for the growth rate of measured GDP when $z > z^*$ versus when $z < z^*$. Recall that measured GDP has a U-shape transitional dynamics (panel (8) in Figure 3). This leads to a decrease in the growth rate of measured GDP over time no matter $z$ is higher or lower than $z^*$. As a result, physical capital heterogeneity tends to generate positive growth-inequality relationship along a transition when $z$ is rising ($z < z^*$) and negative growth-inequality relationship along a transition when $z$ is falling. The reverse is true for human capital heterogeneity.

We summarize our results on transitional dynamics as the following.

1. Consider the growth of full output. If relative capital stock ($K/H$) deviates from its steady state level, then growth and inequality are positively (negatively) related along the transitional trajectory when physical (human) capital heterogeneity dominates.

2. Consider the growth of measured GDP. If relative capital stock ($K/H$) is lower than its steady state level, then growth and inequality are positively (negatively) related along the transitional trajectory when physical (human) capital heterogeneity dominates. If relative capital stock ($K/H$) is higher than its steady state level, then growth and inequality are negatively
related along the transitional trajectory when physical (human) capital heterogeneity dominates.

5 Conclusion

The paper develops a dynamic general equilibrium model with two sectors, multi-dimensional heterogeneity, and joint determination of growth and inequality. We use this framework to analyze the growth-inequality relationship both along the balanced growth path when fundamentals change and along the transitional trajectory as the economy converges to the balanced growth path.

We summarize our results here. (1) There is a common economy-wide growth rate, which is independent of the initial distribution of physical and human capital holdings. (2) Fundamental changes affect income inequality by influencing cross-sector labor allocation. (3) Physical and human capital heterogeneity play opposite roles in the way fundamental changes affect growth-inequality relationship. (4) Growth-inequality relationship along transitional trajectory may differ from the one along the balanced growth path, but relative importance of two types of heterogeneity matters in all cases. In short, the conclusion is that ambiguity of empirical evidence on growth-inequality relationship should not be too surprising.

Our theoretical results provide important implications for empirical studies on the growth-inequality relationship. An unambiguous growth-inequality relationship can only be obtained when it is conditioned on the relative magnitude of physical and human capital heterogeneity. Countries with relatively more physical capital may exhibit different growth-inequality relationship than countries with relatively more human capital. We should also distin-
guish developed and developing countries since former ones are likely to be along the balanced growth path while latter ones are more likely to be long transitions.

This paper considers how fundamental changes affect growth and income inequality. The framework developed here provides a convenient starting point for various extensions. Future work may include studying growth-inequality relationship in response to macroeconomic policies such as tax and monetary policies.
References


Appendix:

1. Proof of Proposition 2.

First note that the growth rate $\psi$ is positively related to $b$:
\[ \frac{\partial \psi}{\partial b} = \theta b^{\theta-1} > 0, \]
and $\rho$ is positively related to $u$ when $\alpha_k > \beta_k$. This allows us to only examine how $b$ and $u$ are affected by fundamental changes. The system of balanced growth path (30) implies $\dot{c}/c = \dot{H}/H$. Then the two equations $\dot{u}/u = 0$ and $\dot{c}/c = \dot{H}/H$ can be rewritten as (after we substitute all other variables)

\[(\theta - 1) (b^\theta - \delta_h) + \theta \delta_h + A \alpha_k \omega^{\alpha_k - 1} - \delta_k - Z^e (1 - \beta_k) \eta \omega^\beta \theta^{\theta - 1} = 0, \quad (A1)\]
\[A \alpha_k \omega^{\alpha_k - 1} - \delta_k - \beta = (1 - \gamma) (b^\theta - \delta_h), \quad (A2)\]

where $\omega \equiv \frac{u}{z}$ and $\eta \equiv \left[\frac{(1-\alpha_k) \beta_k}{\alpha_k(1-\beta_k)}\right]^\beta_k$. Differentiate (A1) and (A2) with respect to $Z^f$ and $Z^e$, we have

\[\frac{\partial b}{\partial Z^f} = \frac{\beta_k \frac{r}{Z^f} \frac{w}{p}}{D} > 0, \]
\[\frac{\partial b}{\partial Z^e} = \frac{(1 - \alpha_k) \frac{r}{Z^e} \frac{w}{p}}{D} > 0, \]

where $D \equiv \left[(1 - \theta) \left(\frac{w}{p}/b\right) + (\theta - \gamma)\right] (1 - \alpha_k) r + (1 - \gamma) \beta_k \frac{w}{p} \theta^{\theta - 1}$.

Recall that $b$ can be written as
\[b = \phi \eta (1 - u) \omega^\beta_k. \quad (A3)\]

Differentiating (A3) with respect to $Z^f$ and $Z^e$ yields

\[\frac{\partial u}{\partial Z^f} = \frac{\beta_k (1 - \beta_k) \frac{r}{Z^f} \frac{w}{p} - \theta (\gamma \delta_h + \beta)}{D}, \]
\[\frac{\partial u}{\partial Z^e} = \frac{(1 - \alpha_k) (1 - \beta_k) \frac{r}{Z^e} \frac{w}{p} - (\gamma \delta_h + \beta)}{D} / Z^e. \]

Both are positive when $\gamma \delta_h + \beta < 0$. Thus we have $\frac{\partial \psi}{\partial Z^f}, \frac{\partial \psi}{\partial Z^e} > 0$, $\frac{\partial \rho}{\partial Z^f}, \frac{\partial \rho}{\partial Z^e} > 0$.

Thus, $\frac{\partial^2 \psi}{\partial Z^f^2}, \frac{\partial^2 \psi}{\partial Z^e^2} > 0$ when $\sigma^2_k$ dominates, and $\frac{\partial^2 \psi}{\partial Z^f^2}, \frac{\partial^2 \psi}{\partial Z^e^2} < 0$ when $\sigma^2_k$ dominates.

Similarly, we calculate

\[\frac{\partial b}{\partial \beta} = - \left[\frac{(1-\alpha_k)r}{\theta b^{\theta - 1}} + \beta_k \frac{w}{p}\right] \frac{1}{D} < 0, \]
\[\frac{\partial b}{\partial \gamma} = \psi \left[\frac{(1-\alpha_k)r}{\theta b^{\theta - 1}} + \beta_k \frac{w}{p}\right] \frac{1}{D} > 0, \]

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and
\[
\frac{\partial u}{\partial \beta} = \frac{1-\beta_k}{w/p} \left[ \beta_k (1-\theta) b + \beta_k \frac{w}{p} \theta + \frac{(1-\alpha_k)r}{\theta^{\rho-1}} \right] D > 0,
\]
\[
\frac{\partial u}{\partial \gamma} = -\psi \frac{(1-\beta_k)}{w/p} \left[ b \beta_k (1-\theta) + \beta_k \frac{w}{p} \theta + \frac{(1-\alpha_k)r}{\theta^{\rho-1}} \right] D < 0.
\]
So (ii) follows. (iii): Follow the same procedure, we obtain
\[
\frac{\partial b}{\partial \alpha_k} = \frac{r}{1-\gamma} \left( \frac{1}{\alpha_k} + \ln \omega \right) \frac{1}{D} > 0,
\]
\[
\frac{\partial b}{\partial \beta_k} = \frac{1}{D} > 0.
\]
Since \(\alpha_k\) and \(\beta_k\) have direct distributional effects, we need to calculate how \(\rho\) changes when \(\alpha_k\) or \(\beta_k\) increases.
\[
\frac{d\rho}{d\alpha_k} = \frac{(1-\beta_k) \left[ (1-\beta_k) u + (1-\alpha_k) (\alpha_k - \beta_k) \frac{du}{d\alpha_k} \right]}{\left[ (1-\alpha_k) + (\alpha_k - \beta_k) u \right]^2},
\]
\[
\frac{d\rho}{d\beta_k} = \frac{(1-\alpha_k) \left[ (1-\alpha_k) (1-u) + (1-\beta_k) (\alpha_k - \beta_k) \frac{du}{d\beta_k} \right]}{\left[ (1-\alpha_k) + (\alpha_k - \beta_k) u \right]^2}.
\]

2. Proof of Proposition 3.
It is easy to show that
\[
\frac{\partial u}{\partial Z^f} = \frac{\partial u}{\partial Z^r} = \frac{(1-\beta_k)}{w/p} \left( \theta \psi + \beta_k \right),
\]
\[
\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial \gamma} = \frac{1-\beta_k}{w/p} \frac{\beta_k (1-\theta) b + \beta_k \frac{w}{p} \theta + \frac{(1-\alpha_k)r}{\theta^{\rho-1}}}{\left[ \frac{(1-\alpha_k)r}{\theta^{\rho-1}} + \beta_k \frac{w}{p} \right]}.
\]
And when \(\gamma = -1\),
\[
\frac{\partial u}{\partial \beta} - \frac{\partial u}{\partial Z^f} = \frac{(1-\alpha_k)r}{\theta^{\rho-1}} (1 + \theta) \left( 1 - \frac{b}{w/p} \right) + 2\beta_k \theta \left( \frac{w}{b} - 1 \right) > 0.
\]
Figure 1: Growth-inequality relationship along BGP ($\sigma^2_k$ dominates).
Figure 2: Growth-inequality relationship along BGP ($\sigma^2_h$ dominates).
Figure 3: Transitional Dynamics
Figure 4: Time paths for the growth rate of full output and income inequality.
Figure 5: Time paths for the growth rate of measured GDP and income inequality.