More on the Limits to Redistribution

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Abstract

The relation between income tax schedules and political processes has attracted attention for a number of years. The existing literature has focussed on the conditions under which a voting mechanism would be decisive within some class of tax schedule, and what sort of redistributive outcome would emerge. This literature has shown that a stable voting equilibrium exists under quite weak restrictions on preferences, and, given the usual property that the income distribution is skewed to the right, that the chosen schedule is usually progressive, in the sense that the average rate rises with income. The present paper sets out to compare the schedule chosen by such a voting mechanism with what would have been chosen by a social planner. In particular, it seeks to recover the implicit welfare function that would lead the planner to choose the same schedule as the median voter, or, symmetrically, what the percentile location of the decisive voter would have to be for the voting equilibrium to deliver the planner’s choice.

1 Introduction

There has been continued interest in the relation between income tax schedules and political processes for many years. One important strand of this literature follows directly from the work of Mirrlees (1971). Whereas he studied the tax schedule that would be chosen by a planner who was concerned to maximize social welfare, a number of subsequent contributions explored what would be the outcome to the same problem if a voting mechanism were invoked instead. Early examples of this alternative approach include Romer, 1975, Roberts, 1977, and Meltzer and Richard, 1981. These papers invariably restricted attention to a simpler tax schedule than the general non-linear schedule studied by Mirrlees, and focussed on a linear tax. For the most part, they also restricted attention to simple majority voting, though the idea was addressed that the voting population might not coincide with the tax-paying population.

Despite these simplifications, the results were sometimes complex, depending on what restrictions were imposed. The traditional approach to ensure that majority voting has a well-defined solution was to assume that preferences were single-peaked. It emerged from the preliminary studies that this could not be assumed in the tax problem, even if all the underlying functions were well-behaved. It then seemed difficult to obtain robust conclusions.
Related work showed that a less demanding condition, variously formulated, but generalized as requiring single-crossing preferences, is sufficient to ensure that a stable voting equilibrium exists (Roberts 1977, Hemming and Keen 1982, Gans and Smart, 1996). Given this condition – which seems fairly weak – there is, inter alia, a majority voting equilibrium, and the median voter is decisive.

Subsequent papers have explored extensions to include modifications to the voting mechanism, inclusion of public provision of public goods, and of publicly provided private goods (Boadway and Marchand, 1995, Corneo and Gruner, 2000, Harms and Zink, 2003, Borck, 2005).

What this literature has neglected, in the main, is any comparison between the outcome driven by the assumed political process and that obtaining under a benevolent planner. The only exception to this has been the occasional comparison between the marginal tax rate chosen (for a linear tax) by the median voter and that which would be chosen by a planner with a Rawlsian objective of maximizing the welfare of the worst-off. It is useful, but hardly surprising, to learn that the marginal rate chosen by the median voter is lower than that chosen by the Rawlsian planner.

The objective of the present paper is to explore the relation between the voting equilibrium and the planning equilibrium more systematically. In particular, what sort of welfare function would imply the type of tax function that would emerge from a majority vote?

The paper examines these questions in respect of two types of tax design that might be chosen by a planner, or offered to a voter – a linear tax and a log-linear tax. It also pays some attention to the way in which the government net revenue requirement may influence these relations.

One interesting result is that, for the types of assumption that commonly underlie this type of analysis, there may be a close correspondence between the welfare maximising and the voting equilibria. Consider, for example, the case of a log-linear tax schedule, a lognormal skill distribution, and logarithmic preferences over consumption and leisure. Then a planner maximising a Benthamite welfare function would choose exactly the same degree of progression as the median voter. This example can be extended to the case of any skill distribution that is symmetric in the logarithm of skill; obviously that includes the lognormal itself, but also a wider class, such as the Champernowne distribution. What is more, it would hold approximately for a wide variety of empirically well-fitting distributions which depart from the lognormal in virtue of having (more or less symmetrically) thicker tails.

The paper then goes on to explore how the relation between the two types of schedule may vary as the assumptions are relaxed. These relaxations include a shift away from the assumption that individuals are completely selfish, so that the planner’s preference for redistribution may be more or less rooted in the underlying preferences of individuals.

There are two common stories on the limits to redistribution. The first is that incentive constraints may restrict an egalitarian planner from achieving as complete a redistribution as the planner would desire. The second is that the political process – typically as modelled by majority voting – may render those considerations irrelevant, with the self-interest of the median voter determining the outcome. This paper explores the relation between these two types of restriction on redistribution.
2 Model set-up

The model is static and deterministic. Hence there is no saving and no risk. No distinction is made between potential workers and the population at large. The production technology follows the radical simplifying assumptions introduced by Mirrlees in his 1971 paper. Workers differ in their skill levels (productivity), but labour inputs of different types are perfect substitutes for each other. A more productive worker simply provides a bigger quantum of homogeneous labour input than a less productive one. Skill is indexed by $n$ which is distributed according to the density function $f(n)$. Aggregate output ($Z$) is proportional to the aggregate quantity of this homogeneous input. Hence if an "$n$-worker" provides a fraction $y(n)$ of the available unit time, we have the worker's gross income $z = ny(n)$, and:

$$Z = \int ny(n)f(n)dn$$

integrated over the support of $f(n)$. It also follows that all income is labour income. $x$ is equivalently consumption and net of tax income, $x = c\{ny(n)\}$, where $c\{\cdot\}$ is the residual income function determined by the tax schedule. Aggregate consumption is:

$$X = \int x(n)f(n)dn$$

The government has a net revenue requirement, $R$, which does not enter personal preferences, and is scaled to aggregate output, $R = rZ$. Hence:

$$X = Z - R = (1 - r)Z$$

2.1 Preferences

Preferences are taken to be quasi loglinear in consumption, so that:

$$u = \alpha \ln x + g(1 - y)$$

where $g' > 0$ and $g'' \leq 0$, and $(1 - y)$ is leisure.

In the absence of taxation, so that $x(n) = ny(n), this form of utility function yields a labour supply implicitly defined by $g'(1 - y) = \alpha g^{-1}$, which is independent of $n$. Income and substitution effects offset each other exactly, and the labour supply function has zero uncompensated elasticity with respect to the wage. This conforms fairly well with the stylized facts, so this is quite an appealing property.

A popular specialisation of the form is to choose $g(\cdot)$ to be $\ln(\cdot)$, yielding Cobb Douglas preferences. This specification is sometimes criticised for embodying an implausibly high (unitary) elasticity of substitution between consumption and leisure, and a CES function with lower elasticity parameter is chosen instead. The disadvantage with this approach is that the chosen parameter controls not only the degree of substitutability, but also the shape of the labour supply function. For example, an elasticity below 1 implies that labour supply is backward bending, more severely so as the elasticity falls. With the quasi loglinear function the elasticity of substitution can be reduced by altering
the form of \( g(\cdot) \), without sacrificing the invariance of labour supply to wage rate changes.

For the classes of tax systems considered, these preferences satisfy the single crossing property. This means that a median voter would be decisive over a choice of tax schedules within each class.

2.2 The skill distribution

Earned income distributions, and the skill distributions that underlie them, are skewed to the right. In this paper, it is assumed that the extent of this skewness is captured by the density of skilled workers at, say, \( n^*/\mu \) coinciding with the density at \( \mu/n^* \), where \( \mu \) is median skill. In other words, the skill distribution will be taken to be symmetric in the logarithm of skill. In the absence of taxes, under quasi loglinear preferences, this would ensure that the distribution of earned income was the same, apart from a constant of proportionality. Hence this would be symmetric in the logarithm of income. The most obvious example of this type of distribution is the lognormal:

\[
n \sim \Lambda(\mu, \sigma)
\]

where \( \mu \) is median skill and \( \sigma^2 \) is the variance of the logarithm of skill, so that the mean skill level is \( \mu e^{\sigma^2/2} \).

However, there are other distributions that exhibit this property. For example, suppose that skill is distributed following the two parameter variant of the Champernowne distribution which has the cumulative form:

\[
F(n) = \tan^{-1} \left( \frac{(\mu/n)^\theta}{\pi/2} \right)
\]

where \( \mu \) is again median skill. This distribution, like the lognormal, is symmetric in the logarithm of income, but has fatter tails\(^1\). The upper tail is asymptotic to the Pareto distribution and \( \theta \) is a parameter related to the Pareto exponent. Lower values of \( \theta \) imply thicker tails.

2.3 The voting mechanism

Simple majority voting is assumed. The obvious objection that voters do not in practice get to vote directly over the tax schedule is not in fact fatal to this approach. Some political scientists (for example, Whiteley et al, 2006) have argued that actual voting is driven more by valence issues than by positional ones. In other words, political parties do not compete by occupying very different positions with respect to the size of the state, whether or not there should be sound economic management, whether or not there should be a substantial provision of public education, whether or not the tax system should be progressive, and so on. Instead, they may present rather similar programmes, and what is at stake is their perceived competence in delivering them. From this

\(^1\) These two properties are shared by the three parameter version of the distribution. Other popular functional forms for fitting to income distributions are the Maddala-Singh and Dagum distributions. Both of these have specific parameterizations that are symmetric in the logarithm (as well as again having an upper tail asymptotic to the Pareto distribution). Indeed when this restriction is made, these two distributions coincide.
perspective, what gives leverage to the preferences of the median voter is that the various parties are forced to offer programmes that accord fairly closely with these preferences, at least in most major dimensions. Programmes that differ too sharply in too many respects from this implied position are not offered by parties with real prospects of gaining or sharing power.

2.4 The welfare function

The government’s objective respects individual utilities. The issue is over how these are to be aggregated. Let the cardinalisation of the utility function which just satisfies the concavity requirement be \( u \). Suppose that transformations of \( u \) are restricted to the isoelastic class. Consider the welfare function:

\[
W = \int \frac{[u(n)^{1-\eta} - 1]}{1-\eta} f(n) dn
\] (7)

for \( \eta > 0 \).

Then \( \eta \) is open to three interpretations, or some combination of them. First, it could reflect the cardinalisation of utility as perceived by individuals themselves. Then \( W \) is simply a Benthamite summation of individual utilities. Second, the perception of utility could be ordinal, so that there is no natural cardinalisation. Then \( \eta \) represents a social choice over inequality aversion. Third, \( \eta \) could be a composite, reflecting the combined effect of the extent of strict concavity in the utility function, coupled with an additional social aversion to inequality.

Notice that, except under the first interpretation, this formulation implies that the inequality aversion is a characteristic of some social planner and is not derived from individual preferences at large. In these two cases, reference will be made to the planner’s welfare optimum to distinguish this from an individualistic welfare optimum.

Suppose instead that any inequality aversion is derived from individual preferences, and that these preferences take the simple additive form:

\[
U(n) = \frac{[u(n)^{1-\nu} - 1]}{1-\nu} + \beta(n) \int \frac{[u(m)^{1-\eta} - 1]}{1-\eta} f(m) dm
\] (8)

where \( \eta > \nu \) implies aversion to inequality in utility. Making the heroic assumption that \( \beta(n) = \beta \), all \( n \), (as well as the equally heroic assumption that inequality aversion is uniform), then Benthamite welfare becomes:

\[
W = \int \frac{[u(n)^{1-\nu} - 1]}{1-\nu} f(n) dn + \beta \int \frac{[u(n)^{1-\eta} - 1]}{1-\eta} f(n) dn
\] (9)

The question under investigation is how the welfare optimising choice of tax schedule is related to the schedule that would be chosen by the median voter. For the present, attention is restricted to the simpler representation of welfare as in 7. Consideration of 9 is deferred until section 5.

\^2 As \( \eta \to 1 \), \( W \to \int \ln [u(n)] f(n) dn \)
3 A loglinear tax schedule - the general case

The government restricts attention to the class of tax schedules with a constant elasticity of net income with respect to gross income:

\[ x(n) = A(ny(n))^a \]  \hspace{1cm} (10)

While chosen for its convenience, this functional form fits remarkably well to the tax schedules that used to be implemented in many countries. These were often characterised by a substantial sequence of tax bands, with marginal rates rising quite frequently and in relatively small steps. More recently, steps to simplify the schedule, resulting in fewer bands and larger steps have meant that it no longer provides such a good fit.

Given quasi loglinear preferences, optimisation now yields:

\[ g' = aa'y^{-1} \]  \hspace{1cm} (11)

which is again independent of \( n \). It is also independent of \( A \), so that a change in the proportional net revenue requirement leaves labour supply unaffected. Totally differentiating 11 yields:

\[ \frac{dy}{da} = \frac{\alpha}{g' - yg''} \]  \hspace{1cm} (12)

and \( g'' \leq 0 \). Hence an increase in progressivity (a fall in \( a \)) must lower labour supply, but it does so uniformly. The (vertical) labour supply function simply shifts back to the left.

With these preliminaries, now consider the relation between the two tax parameters, \( A \) and \( a \). From the government budget constraint 3, we require:

\[ \int [ny(n) - A(ny(n))^a] f(n)dn = r \int ny(n)f(n)dn \]  \hspace{1cm} (13)

Hence:

\[ A = \frac{(1 - r) \int ny(n)f(n)dn}{\int (ny(n))^a f(n)dn} = \frac{(1 - r)y^{1-a} \pi}{\int n^a f(n)dn} \]  \hspace{1cm} (14)

where use is made of the result that \( y(n) = y(a) \), all \( n \).

3.1 The individual’s preference over tax progressivity

Individual utility of an ‘\( n \)-person’, conditioned on \( a \), is then:

\[ u(n) = \alpha [\ln A + a \ln n + a \ln y] + g(1 - y) \]

\[ = \alpha \left[ \ln(1 - r) + \ln \pi + \ln y + a \ln n - \ln \left\{ \int n^a f(n)dn \right\} \right] + g(1 - y) \]  \hspace{1cm} (15)

Differentiating this with respect to \( a \) and setting the resulting expression equal to zero characterises the value the progressivity parameter would take.
if \( n \)-persons were able to choose it to maximise their own utility. Hence for individuals with \( n = n^* \) this condition is:

\[
\frac{d}{da} \left[ \ln \left\{ \int n^a f(n)dn \right\} \right] - \left[ y^{-1} - g'/\alpha \right] \frac{dy}{da} = \ln n^*
\]

which, on substituting from 11 and 12, yields the implicit condition:

\[
\frac{d}{da} \left[ \ln \left\{ \int n^a f(n)dn \right\} \right] - \frac{(1 - a)\alpha}{(g' - yg'')y} = \ln n^*
\]  

Notice that this condition is independent of the particular cardinalisation of the utility function chosen. In the case of majority voting, with the median voter decisive, the condition is:

\[
\frac{d}{da} \left[ \ln \left\{ \int n^a f(n)dn \right\} \right] - \frac{(1 - a)\alpha}{(g' - yg'')y} = \ln \mu
\]  

3.2 Planner’s preference over tax progressivity

Taking the simpler of the two welfare formulations, 7:

\[
W = \int \frac{[u(n)^{1-\eta} - 1]}{1 - \eta} f(n)dn
\]

\[
= \int \frac{\{x(n)^{\alpha_0 + (1-\psi)}\}^{1-\eta} - 1}{1 - \eta} f(n)dn
\]

Substituting from 10 and 14, and differentiating with respect to \( a \), welfare is maximised when the following condition is satisfied:

\[
\int \left[ \frac{d}{da} \left( \ln \left\{ \int n^a f(n)dn \right\} \right) - \frac{(1 - a)\alpha}{(g' - yg'')y} - \ln n \right] n^a(1-\eta)f(n)dn = 0
\]

If the planner’s cardinalisation has \( \eta = 1 \), 20 reduces to:

\[
\frac{d}{da} \left[ \ln \left\{ \int n^a f(n)dn \right\} \right] - \frac{(1 - a)\alpha}{(g' - yg'')y} = \ln n f(n)dn
\]

Recall that \( f(n) \) is symmetric in \( \ln n \) about the central value \( \ln \mu \). Set \( \psi = (\ln n - \ln \mu) \), and let \( t(\psi) \) be the associated density function. Notice that this is symmetric in \( \psi \) and has zero mean. The relation between density functions of transformed random variables is \( f(n) = t(\psi) d\psi/dn \). Hence:

\[
\int \ln n f(n)dn = \int (\psi + \ln \mu) t(\psi)d\psi
\]

\[
= \bar{\psi} + \ln \mu
\]

\[
= \ln \mu
\]  

\( \mu \) is identified as the median skill level. Using 24, 21 becomes:

\[
\frac{d}{da} \left[ \ln \left\{ \int n^a f(n)dn \right\} \right] - \frac{(1 - a)\alpha}{(g' - yg'')y} = \ln \mu
\]
3.3 Condition for the median voter’s choice to coincide with the planner’s optimum

Comparing 20 and 17, when would the choice of the median voter coincide with the planner’s welfare optimum? It is apparent that 25 coincides with the median voter outcome, 18. This means that planners could only achieve their redistributional goals under a majority voting system if those goals were represented by the cardinalisation $\eta = 1$. Planners with more redistributive preferences would be prevented from implementing them by a political process that respected the median voter’s preferences, that is where the median voter is the "fulcrum" of the process. Only if this fulcrum was shifted further down the distribution could a more redistributive system be achieved. However, most discussions of the relevance of the median voter in real political systems tend to shift this fulcrum up not down, reflecting reduced levels of participation lower in the distribution. Accordingly, if the planners’ preferred cardinalisation had $0 < \eta < 1$, they would prefer less redistribution than would be chosen by the median voter, but might be able to implement their preferences if participation was skewed in this way.

It remains to consider the relation between the two optima when the planners’ preference is to implement a tax system that would maximise the sum of utilities as they are experienced by individuals. In other words, the planners are not trying to impose their own personal preferences, but simply doing their best to reflect those of individuals collectively. The point could be underlined by stipulating that individuals had no particular preferences over distributional issues, but subscribed to the view that maximisation of the sum of utilities was an appropriate goal for government. This could be represented, for example, by the welfare formulation 9 but with $\eta = \nu$.

Consider the utility function:

$$u(n) = \frac{\left\{x(n)^\alpha e^{(1-y)}\right\}^{1-\nu} - 1}{1-\nu}$$  \hspace{1cm} (26)

Then the elasticity of marginal utility of income, $\epsilon$, is given by:

$$\epsilon = \frac{xu_{n1}}{u_1} = -[\alpha(\nu - 1) + 1]$$  \hspace{1cm} (27)

Recent work by Layard et al (2006) estimates $\epsilon$ at $-1.3$. Since $\alpha > 0$, this requires $\nu > 1$. For example, $\alpha = 1/2 \implies \nu = 1.6$; or, $\alpha = 2/3 \implies \nu = 1.45$. Hence a Benthamite welfare function, summing over utilities with the cardinalisation experienced by individuals, would require a greater degree of redistribution than will be permitted by the median voter. Only if $\epsilon = 1$, so that $\nu = 1$, which in the present context would imply $\eta = 1$ also, would the two optima coincide. In general, they will not. The paradox here is that the majority voting system delivers an "undemocratic" result, in the sense that there would be universal acknowledgement that the planner’s optimum was objectively preferable for "everybody else". A tax system derived by this sort of political process could be characterised as maximising an implicit welfare function with a particular cardinalisation that would bear only an arbitrary relation to the cardinalisation in the "true" welfare function.
4 A loglinear tax schedule with a lognormal skill distribution

While it is possible to establish the implicit welfare function attendant on majority voting, further exploration of the extent to which the resulting tax system departs from welfare optimality requires analysis of more restricted functional forms. For the present, it will continue to be assumed that the tax schedule is loglinear. As to the skill distribution, for analytic convenience, attention will now be restricted to the lognormal distribution. Finally, several specific variants of the leisure function \( g(\cdot) \) will be analysed. In each case, the formulation of utility is in terms of the cardinalisation that is just concave, so that the associated Hessian matrix is negative semi definite.

With the lognormal distribution, it is possible to integrate the term:

\[
\int n^a f(n) dn = \mu^a e^{a^2 \sigma^2 / 2}
\]

Hence the expression for \( A \) becomes:

\[
A = (1 - r) y^{1 - a} \mu^{1 - a} e^{(1 - a^2) \sigma^2 / 2}
\]

and that for \( x(n) \):

\[
x(n) = (1 - r) y n^a \mu^{1 - a} e^{(1 - a^2) \sigma^2 / 2}
\]

From which:

\[
u(n) = \alpha \ln [x(n)] + g(1 - y)
\]

\[
u(n) = \alpha \ln(1 - r) + \ln y + a \ln n + (1 - a) \ln \mu + (1 - a^2) \sigma^2 / 2 + g(1 - y)
\]

Maximisation of \ref{eq:31} with respect to \( a \) yields, for an \( n^* \) person:

\[
\left( \frac{\alpha}{y} - g' \right) \frac{dy}{da} = \alpha (a \sigma^2 + \ln \mu - \ln n^*)
\]

which for the median individual reduces to:

\[
\left( \frac{\alpha}{y} - g' \right) \frac{dy}{da} = aa \sigma^2
\]

Now consider a welfare function along the lines of \ref{eq:19}, repeated here for convenience:

\[
W = \int \left\{ \frac{x(n)^{\alpha} e^{\alpha(1 - y)}}{1 - \eta} \right\} f(n) dn
\]

Maximisation of this with respect to \( a \) yields:

\[
\left( \frac{\alpha}{y} - g' \right) \frac{dy}{da} = aa \sigma^2 \left[ 1 + \alpha(\eta - 1) \right]
\]
As in the more general case, 34 and 36 coincide when $\eta = 1$. Exploring what happens when $\eta \neq 1$ requires further restrictions on the form of the utility function. Three are considered here in turn. In each case, use is made of the relations 34 and 36. For concreteness, the value of $\sigma$ will be maintained at 0.5 throughout.

### 4.1 Utility linear in leisure

The just-concave cardinalisation of this is:

$$u = \alpha \ln x + (1 - y)$$  \hspace{1cm} (37)

Under the log-linear tax, labour supply is chosen as:

$$y(n) = aa$$  \hspace{1cm} (38)

and:

$$g' = 1, g'' = 0, \frac{dy}{da} = \alpha$$  \hspace{1cm} (39)

Hence the $n^*$ voter’s choice is the solution to:

$$(1 - a) = a(a\sigma^2 + \ln \mu - \ln n^*)$$  \hspace{1cm} (40)

The median voter’s choice is the solution to:

$$(1 - a) = a^2\sigma^2$$  \hspace{1cm} (41)

and that of the planner is the solution to:

$$(1 - a) = a^2\sigma^2 [1 + \alpha(\eta - 1)]$$  \hspace{1cm} (42)

With $\sigma = 0.5$, the median voter would choose $a = 0.8284$. Meanwhile, the planner would be restricted to values of $\eta \geq 1$ if the concavity requirement is to be satisfied. Obviously, if $\eta = 1$, the same $a$ would be chosen. What rank of voter would have made the same choice from self interest as a planner with $\eta > 1$? Equating the right hand sides of 40 with that of 42, we obtain:

$$\ln(n^*/\mu) = -a\alpha(\eta - 1)\sigma^2$$  \hspace{1cm} (43)

Suppose $\alpha = 0.5$ (so that labour supply, in the absence of redistributive taxation, would also be 0.5). Then if the planner’s value of $\eta$ is 2 (3), the choice of $a$ would be 0.7749 (0.7321). This would coincide with a decisive voter at $n^*$ from 43, if $n^*/\mu = 0.9077 (0.8327)$. To put these results in perspective, they would require the decisive voter to be at the 42nd (36th) percentile of the distribution.

### 4.2 Utility loglinear in leisure

The just-concave cardinalisation of this is:

$$u = x^\alpha(1 - y)^{1-\alpha}$$  \hspace{1cm} (44)

Under the log-linear tax, labour supply is chosen as:

$$y(n) = \frac{aa}{1 - \alpha(1 - a)}$$  \hspace{1cm} (45)
and:
\[ g' = 1 - \alpha(1 - a), \quad g'' = -\frac{[1 - \alpha(1 - a)]^2}{(1 - a)^3}, \quad \frac{dy}{da} = -\frac{\alpha(1 - \alpha)}{[1 - \alpha(1 - a)]^2} \tag{46} \]

Hence the \( n^* \) voter’s choice is the solution to:
\[ \frac{(1 - a)(1 - a)}{[1 - \alpha(1 - a)]} = a(\sigma^2 + \ln \mu - \ln n^*) \tag{47} \]

The median voter’s choice is the solution to:
\[ \frac{(1 - a)(1 - a)}{[1 - \alpha(1 - a)]} = a^2 \sigma^2 \tag{48} \]

and that of the planner is the solution to:
\[ \frac{(1 - a)(1 - a)}{[1 - \alpha(1 - a)]} = a^2 \sigma^2 [1 + \alpha(\eta - 1)] \tag{49} \]

Suppose \( \alpha = 0.5 \) (so that labour supply, in the absence of redistributive taxation, would again also be 0.5). Then the median voter would choose \( a = 0.7522 \). If the planner’s value of \( \eta \) were variously 0, 1, 2, or 3, the choice of \( a \) would be 0.8384, 0.7522, 0.6940, or 0.6506. Once again, the relation between the welfare optimum and the rank of the voter who would have to be decisive to achieve this is given by:
\[ \ln(n^*/\mu) = -a\alpha(\eta - 1)\sigma^2 \tag{50} \]

Hence the relative position of the decisive voter would have to be given by \( n^*/\mu \) equal to 1.1105, 1.09169, and 0.8499 respectively. Put differently, these welfare maximising choices would only be supported by a voting mechanism if the decisive voter was located at the 58th, 50th, 43rd, and 36th percentiles respectively.

### 4.3 Utility isoelastic in leisure

Now consider the utility function:
\[ u = x^\alpha e^{g(1 - y)} \tag{51} \]

where \( g(\cdot) \) is isoelastic in leisure with an elasticity of \(-2\), i.e. \((1 - y)g''/g' = -2\). This has the consequence of reducing the elasticity of substitution between consumption and leisure substantially below the unit value implied by the loglinear formulation. The just-concave form has the parameterisation:
\[ g = -\frac{4(1 - \alpha)^2}{(2 - \alpha)(1 - y)} \tag{52} \]

Hence:
\[ g' = -\frac{4(1 - \alpha)^2}{(2 - \alpha)(1 - y)^2} \tag{53} \]

For comparability with the previous illustrations, suppose that in the absence of taxation, \( y = 0.5 \). This requires that \( \alpha = 2/3 \). Substituting into the labour supply optimisation equation 11 yields a quadratic:
\[ 2a(1 - y)^2 = y \tag{54} \]
which has the solution:

\[ y = \frac{1 + 4a - \sqrt{1 + 8a}}{4a} \]  

(55)

Differentiating this with respect to \( a \), and substituting into the common left hand side of the various expressions for optimal choice of \( a \), we obtain, after some manipulation:

\[ \frac{\alpha}{y} - g' \frac{dy}{da} = \frac{\alpha(1-a)(1+8a)^{-1/2}}{a} \]  

(56)

Equating this to \( \alpha a\sigma^2 [1 + \alpha(\eta - 1)] \) yields a polynomial equation with one real root in the welfare maximising choice of \( a \). If the planner’s value of \( \eta \) were variously 0, 1, 2, or 3, the choice of \( a \) would be 0.8377, 0.6929, 0.6156, or 0.5641. As would be expected, given the reduced substitutability between consumption and leisure, progressivity rises more steeply (\( a \) falls faster) as \( \eta \) rises than in the loglinear case. The associated percentiles for a decisive voter would be the 61st, 50th, 42nd, and 35th percentiles respectively.

4.4 Conclusion

A rather interesting pattern emerges from these three sets of results. They confirm, of course, that the median voter would choose the degree of tax progression that would be welfare-optimal if welfare was represented by summation over the logarithmic cardinalisation of utility, i.e. by \( \eta = 1 \). However, there is another type of uniformity as well; the percentile position of the voter who would choose the welfare optimum when these were represented by summation over more concave cardinalisations is much the same whichever form of utility function is involved. Thus, if \( \eta = 2 \), the welfare optimum would have been chosen by voters at the 42nd, 43rd, and 42nd percentiles in the three cases. If \( \eta = 3 \), the welfare optimum would have been chosen by voters at the 36th, 36th, and 35th percentiles. In other words, the optimal degree of progressivity is sensitive to substitutability within the utility function, but the extent of the discrepancy between the majority voting equilibrium and the welfare optimum is not, where this discrepancy is measured by the percentile gap between those who control the voting equilibrium and those who would have chosen the welfare optimum.

5 Altruism

A specific additive isoelastic form of this was introduced at 8 and 9, reproduced here with the restriction of a common \( \beta \):

\[ U(n) = \frac{\left[u(n)^{1-\nu} - 1\right]}{1-\nu} + \beta \int \frac{\left[u(m)^{1-\eta} - 1\right]}{1-\eta} f(m)dm \]

\[ W = \int \frac{\left[u(n)^{1-\nu} - 1\right]}{1-\nu} f(n)dn + \beta \int \frac{\left[u(n)^{1-\eta} - 1\right]}{1-\eta} f(n)dn \]

The assumption is retained that \( f(\cdot) \) is lognormal. Note that the two parameters \( \nu \) and \( \eta \) are common to the two functions; this follows from the additive derivation of \( W \) from \( U \), reflecting the assumption of a planner with an individualistic social welfare function. Assuming that the population is large, the
individual’s own weight in the second term is trivial, so that labour supply is
una
ff
ected by the addition of this term. The individual’s preferred value of
a
is
a
ff
ected, however:
and are each the sum of two terms, with the
second term in common. We can again set out to identify the skill level (n∗) at
which an individual would choose the welfare optimum. Without going through
the detailed and rather tedious derivation, the condition for this is:
\[ \left( \frac{n^*}{\mu} \right)^{\alpha} e^{(1-a^2)\sigma^2/2} \ln \left( \frac{n^*}{\mu} \right) - \alpha a \sigma^2 \] (57)
The solution for any \( \nu \neq 1 \) will be complicated and dependent on the specific
form of \( g(\cdot) \) and the detailed parameterisation. If \( \nu = 1 \), the condition 57 reduces
to \( n^* = \mu \). In other words, the welfare optimality of the median voter choice is
preserved in this altruistic case, provided that the cardinalisation of own utility
is logarithmic in consumption. This places no restriction on the value of \( \eta \).
However, the choice of \( a \) is different from that in the selfish case. Let:
\[ Med(a) = \left( \frac{\alpha}{y} - g' \right) \frac{dy}{da} - \alpha a \sigma^2 \] (58)
and let:
\[ Plan(a) = \left( \frac{\alpha}{y} - g' \right) \frac{dy}{da} - a \sigma^2 [1 + \alpha(\eta - 1)] \] (59)
Recall from the earlier analysis that for a selfish median voter, optimisation of \( a \)
requires \( Med(a) = 0 \) whereas the planner’s optimisation requires \( Plan(a) = 0 \).
Then optimisation in the present altruistic case requires:
\[ Med(a) + \beta \tau(a) Plan(a) = 0 \] (60)
where
\[ \tau(a) = \left( 1 - \tau \right) g \mu e^{[1-a^2] [1+\alpha(\eta-1)] \sigma^2/2 \alpha(1-\eta)} e^{g(\cdot)(1-\eta)} > 0 \] (61)
Optimisation by the altruistic median voter (and equivalently of the associated
individualistic welfare function) now involves a weighted average of the selfish
and planner conditions (assuming that the planner’s inequality aversion coincides
with that of individuals). If individuals are highly altruistic (high \( \beta \), then
the voting equilibrium will lie close to the planner’s optimum previously calculated;
if they are rather selfish (low \( \beta \), it will lie closer to the completely
selfish case. In either case, the impact of altruism and of inequality aversion
is correctly reflected in the voting equilibrium which would underpin a welfare
optimum.

6 A linear tax schedule

The preceding analysis has utilised a loglinear tax schedule, partly because this
offers a reasonable stylized representation of a tax system with rising marginal
rates, partly because of its tractability. In voting as well as other contexts, much
Attention has been paid to linear tax schedules, and these are considered briefly in this section. Net income is now related to gross income by:

\[ x(n) = (1 - t)ny(n) + G \]  \hspace{1cm} (62)

where \( t \) is the constant marginal tax rate and \( G \) is a universal lump sum payment to all workers. With the quasiloglinear utility function, \( u = \alpha \ln x + g(1 - y) \), labour supply is given by:

\[ g' = \frac{(1 - t)n\alpha}{(1 - t)ny + G} \]  \hspace{1cm} (63)

If earnings capacity at the lower end of the skill distribution is sufficiently low, individuals with skill below some level, \( n_0 \), will choose to be idle, setting \( y = 0 \), and \( x = G \). This means that the integration in the government budget constraint is over the skill range \( n \geq n_0 \), where \( n_0 \) is determined by the tax parameters and the form of the utility function. We abstract from this complication, assuming that the lower bound on skill, \( \bar{n} \geq n_0(\text{max}) \), where \( n_0(\text{max}) \) is the highest value of \( n_0 \) that comes into contention. It is convenient to maintain the assumption that skill has a lognormal distribution, but to assume that this is truncated above and below, at values of \( \bar{n} \) and \( \underline{n} \), respectively. It will be further assumed that \( \bar{n}/\mu = \mu/\underline{n} \), so that the truncated distribution remains symmetric in the logarithm of skill.

Attention is restricted to the case where utility is linear in leisure. Hence \( g'(\cdot) = 1 \). Substituting into (63),

\[ y(n) = \alpha - \frac{G}{(1 - t)n} \]  \hspace{1cm} (64)

where

\[ \bar{n} \geq n_0 = \frac{G}{(1 - t)\alpha} \]  \hspace{1cm} (65)

Consumption of an \( n \)-person is

\[ x(n) = (1 - t)ny + G \]  \hspace{1cm} (66)

\[ = (1 - t)n\alpha \]  \hspace{1cm} (67)

Aggregating over the population

\[ X = (1 - t)\alpha \int_{\underline{n}}^{\bar{n}} nh(n)dn = (1 - t)\alpha \pi \]

\[ Z = \int_{\underline{n}}^{\bar{n}} ngy(n)h(n)dn = a\pi - \frac{G}{(1 - t)} \]

Notice that, to ensure that \( \int_{\underline{n}}^{\bar{n}} h(n)dn = 1 \), i.e. that \( h(n) \) remains a density, the standard lognormal density, \( f(n) \), has to be "grossed up" by a factor taking into account the otherwise missing values. Specifically,

\[ h(n) = \frac{f(n)}{\int_{\underline{n}}^{\bar{n}} f(n)dn} \]  \hspace{1cm} (68)
Nor does the adjustment stop there. While, by design, the number of "missing" workers is the same at both ends of the distribution, those at the top are more productive, so the mean skill level is rather more substantially reduced. However neither adjustment is particularly significant. The government budget constraint is

\[ X = (1 - r)Z \]  
\[ (1 - t)\alpha\pi = (1 - r)\alpha\pi - \frac{(1 - r)}{(1 - t)}G \]

Hence

\[ \frac{G}{(1 - t)} = \frac{(t - r)}{(1 - r)}\alpha\pi \]

Substituting into the utility function of an n-person

\[ u(n) = \alpha \ln(1 - t) + \alpha \ln \alpha + \alpha \ln n + 1 - \alpha + \frac{(t - r)\alpha\pi}{(1 - r)n} \]

Differentiating with respect to the tax rate, and equating to zero, this is maximised if the tax rate is set according to

\[ (1 - t) = (1 - r)n/\pi \]

From which

\[ (t - r) = (1 - r)(1 - n/\pi) \]

and

\[ n_0 = \frac{G}{\alpha(1 - t)} = \frac{(t - r)\pi}{(1 - r)} = (\pi - n) \]

In other words, the maximum skill level at which a worker would choose to be idle is equal to the difference between the average skill and that of the decisive voter. For example, if the median voter is decisive, \( n_0 \) is the difference between the mean and the median. Setting the median equal to one by choice of units, and using the uncorrected mean, this would equal \((e^{1.93^2} - 1)\). For the maintained assumption that \( \sigma = 0.5 \), this comes to 0.1331, or 13% of the median. If the correction to the mean estimated in the previous footnote was made, then for \( \bar{n} = 0.2 \) (0.3), \( n_0 \) would fall to 0.1308 (0.1171). Hence the required adjustments are small, and the choice of \( t \) and \( G \) not very sensitive to them.

Now consider the welfare choice where a government adopts the log cardinalisation (\( n = 1 \) in the earlier discussion). In the remainder of the section, the adjustments just discussed are ingored, though the means for making them are noted in the next footnote. Integrating over utilities, welfare is given by

\[ W = \alpha \ln(1 - t) + \alpha \ln \alpha + \alpha \int \ln nb(n)dn + 1 - \alpha + \frac{(t - r)\alpha\pi}{(1 - r)} \int n^{-1}h(n)dn \]

\[ (76) \]

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\[ ^3\text{Consider the integral } I_0 = \int_0^\infty f(n)dn, \text{ where } f(n) = \text{lognormal, } \Lambda(\mu, \sigma), \text{ and the limits are related as in the text. Then using the transformation } \psi = (\ln n - \ln \mu)/\sigma, \text{ we have } I_0 = \text{as a standard normal density integrated between } \psi_0 = (\ln \bar{n} - \ln \mu)/\sigma \text{ and } \psi_1 = (\ln \bar{n} - \ln \mu)/\sigma. \text{ Hence } \psi_1 = -\psi_0. \text{ Now consider the integral } I_1 = \int_0^\infty nf(n)dn. \text{ Using the transformation } \varphi = (\ln n - \ln \mu)/\sigma - \sigma \text{, we have } I_1 \text{ as a standard normal density integrated between } \varphi_0 = (\ln \bar{n} - \ln \mu)/\sigma - \sigma = \varphi_0 - \sigma \text{ and } \varphi_1 = (\ln \bar{n} - \ln \mu)/\sigma - \sigma = \varphi_1 - \sigma. \text{ Hence, } I_1 < I_0. \text{ For example, let } \mu = 1, \text{ and } \sigma = 0.5, \text{ as before. Then for } \bar{n} = 0.2 \text{ (0.3), } I_0 = 0.9987 \text{ (0.9840), and } I_1 = 0.9966 \text{ (0.9700). } 1/I_0 \text{ is the factor that needs to be applied to scale up } f(n); I_1/I_0 \text{ is the factor that needs to be applied to scale down the mean.} \]
In the absence of truncation, \( \int n^{-1} f(n)dn = \mu^{-1}e^{\sigma^2/2} \). In the presence of truncation, the ratio of the corrected to the uncorrected value is identical to that required in adjusting \( \bar{n} \). Hence the adjusted product of the two integrands has the square of this ratio. Even so, it remains very small and is neglected in the remaining discussion. Hence \( W \) is approximately

\[
W = \alpha \ln(1-t) + \alpha \ln \alpha + \alpha \ln \mu + 1 - \alpha + \frac{(t-r)\alpha e^{\sigma^2}}{1-r}
\] (77)

Maximising \( W \) with respect to \( t \) yields the condition

\[
(1-t) = (1-r)e^{-\sigma^2}
\] (78)

Comparing 78 with 73 it appears that even with the log cardinalisation, the welfare optimum would not be chosen by the median voter. It would require the decisive voter to be some way further down the distribution, with a skill \( n^* = \mu e^{-\sigma^2/2} \). For the lognormal, this is the geometric mean of the median and the mode. With \( \sigma = 0.5 \), \( n^* \) would be at the 40th percentile.

At this stage of the analysis, it is unclear why the median voter would choose a relatively less redistributive schedule than the hypothetical planner under a linear tax relative to their respective choices under a loglinear one. Elucidating this requires further work. It is worth noting, however, what the median voter would do faced with a two stage choice, first between types of tax schedule, and secondly, over the detailed structure within the preferred schedule. For the case considered here, the median voter would choose the loglinear schedule over the linear one. In that sense, the loglinear results are more interesting than the linear ones. However, this conclusion rests on a neglect of the increased administrative costs of more complex structures.

7 Conclusion

For the highly specialised, but not wholly implausible, combination of quasi loglinear preferences, a loglinear tax schedule, and a skill distribution that is symmetric in the logarithm of skill, a robust analytic result can be obtained. It is that a selfish median voter will choose the same parameterisation as would a planner who adopted a logarithmic cardinalisation of welfare. A planner with more redistributive tastes would be unable to obtain sufficient political support. However, in this situation, the legitimacy of this redistributive impulse must be questionable. If, instead, voters have some degree of inequality aversion that is additive to their own direct utility, the voting equilibrium would deliver the optimum that would be chosen by a Benthamite planner, provided own utility had a logarithmic cardinalisation. In these circumstances, the voting process would deliver the happy accident of a welfare-optimising tax regime.

If the loglinear tax schedule is replaced by a linear schedule, analytic results are hard to obtain. On the basis of the results reported here, the median voter appears to choose a less redistributive schedule than the hypothetical planner even when the latter adopts a logarithmic cardinalisation. It is worth noting, however, that the median voter would reject the linear schedule if given the alternative of a loglinear one.
To extend the analysis to more general cases will require numerical simulation. One purpose of the present paper has been to provide an analytical special case against which the results of these simulations can be calibrated.
References


