

Collusion without Patience

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Abstract

Tacit collusion is usually linked to repeated interactions between patient firms. We show that it can also arise in a one-shot duopoly. When firms choose capacities and face outsourcing contracts with minimum order quantities, gaps in feasible outputs allow commitment not to sell intermediate quantities. For a range of parameters, there exists a collusive equilibrium in which both firms produce less and earn more than under competition. Three other equilibria can also arise: competitive, leadership, and miscoordination. We then extend the model to general two-sided limited commitment and show that it yields the same set of equilibria.

JEL: C72, C73, D43, L13

Keywords: collusion, capacity, minimum order quantity, duopoly, Cournot, Stackelberg, limited commitment

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1 Introduction

Collusion is typically associated with repeated interactions among patient firms. In contrast, standard oligopoly models suggest that firms that heavily discount the future are unlikely to sustain collusion, as they are unwilling to incur the short-term costs of punishing deviations. We argue that extending a standard Cournot framework to include the strategic choice between in-house production and outsourcing can help rationalize tacit collusion even among impatient firms. Specifically, the presence of *minimum order quantity* (MOQ) constraints in outsourcing introduces a commitment device, with the potential for making collusive outcomes stable despite limited patience.

Minimum order quantity requirements are common across a range of industries where production and delivery processes benefit from economies of scale, including hardware, textiles, and chemical products. A familiar example is book printing, where nearly every order involves a minimum number of copies to be produced and sold.¹ While MOQs have received considerable attention in the operations management literature (see the literature section), they remain underexplored in the field of industrial organization. We take a first step toward filling this gap by showing that outsourcing under MOQ constraints can serve as a credible threat in the context of a tacit agreement to limit production volumes.

We consider a two-stage Cournot duopoly model. In the first stage, both firms simultaneously and publicly choose their in-house production capacity. In the second stage, each firm decides how many units to offer for sale. A firm can fulfill any volume of sales up to its capacity through in-house production. To sell beyond that level, it must resort to outsourcing. Crucially, outsourcing is subject to an exogenous minimum order quantity: the firm cannot outsource fewer than a pre-specified number of units.²

Our main result shows that if the MOQ is larger than the “Cournot quantity”—the quantity each firm would produce according to a standard Cournot model—then it is an equilibrium for both firms to set capacity below the Cournot quantity and produce up to their capacity. In this equilibrium, both firms—fixing the competitors’ quantity—would have an incentive to increase capacity and produce more. These deviations are prevented by the credible threat of the other firm to resort to outsourcing upon observing an unexpectedly large capacity choice. If the MOQ is not too large, this

¹Sources: qoblex.com, www.lokad.com, and Heese (2025).

²Disposal is free in our model: firms can put on the market a fraction of the units outsourced. We will focus, though, on parameter constellations for which disposal is never optimal.

“collusive” equilibrium is the most profitable one for the firms.

Our model also has other equilibria. In particular, it can be an equilibrium for one firm to outsource a large amount and for the other to produce a small amount in-house. Interestingly, the Cournot quantities are an equilibrium outcome for small and for large MOQs, but not for intermediate ones. For intermediate MOQs, in a “Cournot equilibrium” firms would have to set up sufficient capacity to produce the Cournot quantity in-house. Yet, after such a capacity choice by the competitor, a firm has a profitable deviation to zero capacity. Following this deviation, the only continuation equilibrium has the deviating firm outsourcing and the competitor producing a small amount in-house. We show that this deviation can indeed be profitable.

In our model firms can freely dispose of units of product, yet we focus on parameter values for which it is never profitable to do so. Hence, the capacity-setting stage of the game can be interpreted as the firm voluntarily committing to sell a quantity within a given subset—namely, a quantity not larger than in-house capacity—or not smaller than capacity. In the last part of the paper, we characterize all equilibrium outcomes of the Cournot game that can be rationalized by some form of voluntary, partial commitment. There, we highlight that collusive outcomes in which both firms sell less than the Cournot amount can result from voluntary commitment to sell either little (the on-path outcome) or a lot (the punishment threat for deviations of the competitor), but not intermediate quantities. This gap makes the threat of punishment credible while ensuring that the firm itself will not produce “too much”.

In the following subsection, we review the related literature. The duopoly model with capacity and minimum order quantities is presented in Section 2 and analyzed in Sections 3 and 4. In Section 5 we broaden the scope to consider voluntary partial commitment in a duopoly setting.

1.1 Literature Review

We model oligopolistic competition among firms that can set capacity and outsource production, but are subject to minimum order quantity constraints. A substantial body of literature has studied the use of capacity as a strategic tool to influence competitors. Seminal contributions have examined how incumbent firms use capacity commitments either to deter entry (Spence, 1977) or to assume a Stackelberg leader

position and manipulate the entrant’s best response (Dixit, 1980).³ These models, like ours, are framed within Cournot competition. Kreps and Scheinkman (1983) consider an augmented duopoly game in which the first stage can be interpreted as capacity setting. In their model, firms choose capacities and then compete à la Bertrand, ultimately showing that capacity choice leads to Cournot outcomes. In contrast, we introduce capacity choice to rationalize outcomes *other than* Cournot. Firms choose prices rather than quantities also in Cabon-Dhersin and Drouhin (2014), where capacity setting leads to collusion as the unique outcome.

To the best of our knowledge, the economics literature has not considered the strategic implications of minimum order quantities. These have instead been explored in the operations management literature. Notable contributions include Chintapalli, Disney, and Tang (2017), Tuncel, Taneri, and Hasija (2022) and Heese (2025). Chintapalli et al. (2017) shows how MOQ, combined with advanced order discounts, can be a way to coordinate supply chains. Tuncel et al. (2022) seeks to rationalize the widespread use of MOQ contracts by attributing their popularity to contractual simplicity. Heese (2025) demonstrates that MOQ contracts enable retailers to coordinate on profitable equilibria, in which both firms source exclusively from a single supplier and sustain elevated prices in the final market.

In our model, tacit collusion is sustained by the threat of punishment—in the form of outsourcing large quantities. Tacit collusion has traditionally been associated with repeated interactions among patient firms.⁴ Seminal contributions on this topic include Friedman (1971), Green and Porter (1984), and Rotemberg and Saloner (1986).⁵

In Section 5, we discuss how our model can be interpreted as an instance of bilateral partial commitment. In practice, limited commitment can take various forms. For example, Saloner (1987) and Romano and Yildirim (2005) examine firms that can set a lower bound on their quantities while retaining the option to produce more. Other studies have considered firms that revise their actions—either at fixed

³Additional contributions include Spulber (1981), Bulow, Geanakoplos, and Klemperer (1985), and Dixon (1986).

⁴See Ivaldi, Jullien, Rey, Seabright, and Tirole (2007) for a survey of the economic conditions under which tacit collusion is feasible.

⁵More broadly, folk theorems (Abreu, Pearce, and Stacchetti, 1986; Fudenberg, 1986; Fudenberg, Levine, and Maskin, 1994) establish that, for sufficiently patient players, any feasible and individually rational outcome can be supported as an equilibrium. In those models, patience is crucial, and so is the fact that punishment need not be a Nash equilibrium. In contrast, neither patience nor off-equilibrium punishment is essential in our setting.

intervals (Maskin and Tirole, 1988), stochastically (Kamada and Kandori, 2020), or by incurring adjustment costs (Henkel, 2002; Caruana and Einav, 2008).⁶ A recent literature explores algorithmic pricing as a mechanism through which firms may achieve tacit collusion (Calvano, Calzolari, Denicolò, and Pastorello, 2020; CMA, 2021; Asker, Fershtman, and Pakes, 2022).

In our setting, partial commitment takes the form of commitment to a subset of available actions. Renou (2009) and Bade, Haeringer, and Renou (2009) study bilateral partial commitment defined in this way, focusing on settings in which players can commit to single actions or to subsets of actions. In contrast, we explore, by way of example, the implications of players having access only to a limited set of subsets to which they can commit. In doing so, we build on Bizzotto, Hinnosaar, and Vigier (2025), where one player chooses from a restricted menu of action subsets for commitment.

2 Model

The Setting. We study a duopoly model with linear inverse demand

$$P(Q_1, Q_2) = \max\{0, A - B(Q_1 + Q_2)\},$$

for some $A, B > 0$, where $Q_1, Q_2 \geq 0$ are the quantities sold by firm 1 and 2, respectively. At the onset, firm $i \in \{1, 2\}$ sets its production capacity K_i . Capacity investment is costless. The firm can then produce up to K_i units at a marginal cost $C \in [0, A)$. Else, the firm can outsource. Outsourcing does not require a long-term decision, but is subject to a minimum order quantity $M > 0$: the firm has to pay for at least M units. The minimum order quantity is the same for both firms. Outsourcing quantity Q_i , costs $C \max\{M, Q_i\}$, so – as long as the firm outsources at least M – the unit cost is the same for in-house production and outsourcing.⁷ The game unfolds in two stages:

1. Firm 1 and 2 choose, respectively, capacity $K_1, K_2 \in [0, M]$ simultaneously.⁸

⁶For a broader discussion of commitment devices, including those contingent on other players' commitments, see Kalai, Kalai, Lehrer, and Samet (2010).

⁷All our qualitative results remain unchanged if we assume an increasing cost of capacity or assume the unit cost of outsourcing is different from the unit cost of in-house production.

⁸As in-house production and outsourcing have the same marginal cost, it is without loss of

2. Firms 1 and 2 observe each others' choice of capacity and simultaneously choose quantities $Q_1, Q_2 \in [0, 1]$. We will refer to this stage as subgame (K_1, K_2) .

Firms aim to maximize profit, which, for firm i correspond to:

$$\begin{cases} P(Q_i, Q_j)Q_i - CQ_i & \text{if } Q_i \in [0, K_i] \cup [m, 1] \\ P(Q_i, Q_j)Q_i - CM & \text{otherwise} \end{cases}$$

We assume the marginal cost is large relative to the minimum order quantity and the slope of the demand curve. We will show that this assumption makes outsourcing fewer than M units unprofitable.

Assumption 1. $C \geq BM$.

We focus on pure-strategy subgame perfect equilibria, or just "equilibria" for short. An *equilibrium outcome* is a pair (q_1^*, q_2^*) such that there exists an equilibrium in which firms 1 and 2 choose, respectively, quantity q_1^* and q_2^* .

Rescaling. To simplify the notation, we rescale all quantities (Q_1, Q_2 and M) by $\frac{B}{A-C}$ and monetary amounts (P and C) by $\frac{1}{A-C}$. We use small letters for rescaled values: for instance $q_1 := \frac{BQ_1}{A-C}$ and $p := \frac{P}{A-C}$. The rescaled inverse demand is

$$p(q_1, q_2) = \max\{0, 1 + c - q_1 - q_2\}, \quad (1)$$

and we can restrict attention to $q_1, q_2 \in [0, 1]$;⁹ Assumption 1 takes the simpler form $c \geq m$.

The following terminology will be used extensively. If firms were to compete à la Cournot (as it would happen in subgame (m, m)), they would choose quantities that are best responses to each others', so $q_1 = R(q_2)$ and $q_2 = R(q_1)$, where $R(q) := \frac{1-q}{2}$. We refer to $R(q)$ as the *unconstrained best response* to q . As $R(z) = z \Leftrightarrow z = \frac{1}{3}$, we define $q^C = \frac{1}{3}$, and refer to it as *Cournot quantity*. We also talk of *Cournot market quantity*, $q_T^C = 2q^C = \frac{2}{3}$, *Cournot outcome*, $(q_1, q_2) = (q^C, q^C)$ and *Cournot payoff* $\pi^C = \pi(q^C, q^C) = \frac{1}{9}$.

generality to assume that capacities are not larger than M .

⁹Quantities above 1 would lead to negative profits.

If instead firm i is the only firm present in the market and does not have any capacity constraint (say $m = 1$, and consider the subgame $K_i = 1$ and $K_j = 0$), then the firm would choose the *monopoly quantity* $q^M = \frac{1}{2}$.

3 Preliminary Observations

In this section, we discuss a few general properties of the equilibrium strategies, then consider two benchmarks.

3.1 Equilibrium Properties

No Outsourcing Below m . Our model gives firms the option to outsource and then sell a quantity smaller than the quantity outsourced. For instance, firms can outsource and sell less than the minimum order quantity m . This strategy is never optimal.

Lemma 1. *In all subgames (K_1, K_2) , in equilibrium firms always choose quantities either below their capacity or above the minimum order quantity.*

Figure 1 shows firm i 's profit as a function of q_i when both firms choose the same capacity and firm j produces at capacity. The graph is drawn for $K_i = K_j = \frac{1}{4}$, $c = \frac{1}{2}$ and $m = \frac{1}{2}$. Quantities $q_i \in (K_1, m)$ are clearly suboptimal.

Let $\pi(q_i, q_j)$ denote the profit of firm i for $q_i \in [0, K_i] \cup [m, 1]$ and $q_i + q_j \leq 1$, so that:

$$\pi(q_i, q_j) = (1 - q_i - q_j)q_i.$$

In light of Lemma 1, we refer, with a slight abuse, to π as the profit functions. We instead refer to the correspondence $R_K[0, 1] \rightrightarrows [0, K] \cup [m, 1]$ given by

$$R_K(q) := \arg \max_{x \in [0, K] \cup [m, 1]} \pi(x, q)$$

as the *constrained best response* to q_2 . Since π is continuous and the feasible set is compact, existence of maximizers follows from the Weierstrass Extreme Value Theorem. Since π is strictly concave, there are at most two maximizers.

Subgame Equilibria.

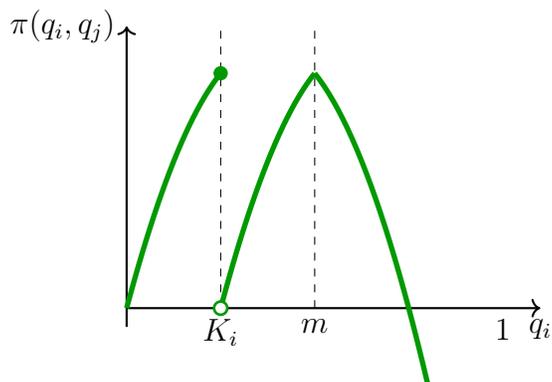


FIGURE 1

Lemma 2. *Every subgame (K_1, K_2) has a Nash equilibrium in pure strategies.*

Proof: Define

$$\begin{aligned}\hat{R}_K(q) &= \max R_K(q); \\ \phi_i(q) &= \hat{R}_{K_j}(\hat{R}_{K_i}(q)), \quad \text{for } i \in \{1, 2\}.\end{aligned}$$

By construction, \hat{R}_K is weakly decreasing, hence ϕ_i is weakly increasing. Since $([0, 1], \leq)$ is a complete lattice and ϕ_i is order-preserving, Tarski's Fixed-Point Theorem guarantees the existence of a fixed point q^* of ϕ_i . Define $q_i^* = \hat{R}_{K_i}(q_j^*)$. Then

$$q_1^* = \hat{R}_{K_1}(q_2^*) \in R_{K_1}(q_2^*), \quad q_2^* = \hat{R}_{K_2}(q_1^*) \in R_{K_2}(q_1^*).$$

Thus (q_1^*, q_2^*) is a Nash equilibrium. ■

3.2 Benchmarks

We discuss two benchmarks. First, we let in-house production be the only option – as is the case when the minimum order quantity is too large for outsourcing to be economically viable. We then let outsourcing be *essentially* the only option—as is the case when minimum order quantities are so small as to make in-house production strategically irrelevant. In both cases, the unique equilibrium outcome is the Cournot one.

Only in-house production. If the minimum order quantity is too large for outsourcing to be viable – say, $m > 1$ – firms have no strategic reason to limit their capacity. It is immediate to verify that the unique quantities associated with an equilibrium are the Cournot ones.¹⁰

Only Outsourcing. When there is no minimum order quantity ($m = 0$), the game reduces to a standard Cournot game. The unique equilibrium outcome is the Cournot one. The Cournot quantity is the unique equilibrium quantity also for a positive, but sufficiently small, minimum order quantity.

Proposition 1. *If $m \leq q^C$, the unique equilibrium outcome is the Cournot outcome.*

The proof is in the Appendix. In the rest of the analysis, we focus on the case $m \in (q^C, 1)$.

4 Analysis

In this section, we characterize all equilibrium outcomes. We start with an equilibrium in which both firms produce in-house quantities smaller than the Cournot ones. As firms are better off than in the Cournot case, we refer to this as a collusive equilibrium.

Proposition 2 (Collusive Equilibrium). *If $m \in (q^C, q_T^C]$, then $(R(m), R(m))$ is an equilibrium outcome. This outcome satisfies $(R(m), R(m)) < (q^C, q^C)$ and ensures higher profits than the Cournot one.*

The following properties will be useful. For any $m > q^C$:

$$R_K(R(m)) = \begin{cases} m & \text{if } K < R(m) \\ \{m, K\} & \text{if } K = R(m) \\ K & \text{if } K > R(m) \end{cases} \quad (2)$$

and

$$R_{R(m)}(q) = \begin{cases} \max\{R(q), m\} & \text{if } q < R(m) \\ R(m) & \text{if } q \in [R(m), m] \end{cases} \quad (3)$$

¹⁰In equilibrium firm select capacities not smaller than q^C .

Note that $q^M \in (q^C, q_T^C]$, and $R(q^M) = \frac{q^M}{2}$. Proposition 2 then establishes that – for $m = q^M$ – there is an equilibrium in which both firms produce the same amount and together they produce the monopoly quantity: $q_1 = q_2 = \frac{q^M}{2}$.

We prove the proposition by showing that any pair of strategies that satisfy the following properties constitute an equilibrium for $m > q^C$.

Stage 1: the firms set $(K_1, K_2) = (R(m), R(m))$;

Stage 2: • on path (i.e. in subgame $(R(m), R(m))$), firms choose $(q_1, q_2) = (R(m), R(m))$;
 • if firm i deviates to $K_i > R(m)$, then firms select $q_i = R(m)$ and $q_j = m$;
 • if firm i deviates to $K_i < R(m)$, then firms select $q_i = m$ and $q_j = R(m)$.

To verify that these strategies constitute an equilibrium, consider first the subgame $(R(m), R(m))$. Figure 2a illustrates for $m = q^M$. Quantities $q_i = q_j = R(m)$ are a pair of best responses (see (2)). Figure 1 shows the function $\pi(q_i, R(m))$ for $m = q^M$.

Suppose firm i deviates in stage 1 to some $K_i > R(m)$. Outcome $q_i = R(m)$, $q_j = m$ is a subgame equilibrium outcome: firm i chooses the unconstrained best response and firm j chooses the constrained best response (see (2)). Figure 2b illustrates. The subgame equilibrium makes the deviation unprofitable.

The argument is similar following a deviation to $K_i < R(m)$. Outcome $q_i = m$, $q_j = R(m)$ is a subgame equilibrium outcome (Figure 2b illustrates), and, also in this case, the deviation is unprofitable (see (2)).

The outcome $(R(m), R(m))$ ensures to each firm a payoff of $\frac{m(1-m)}{2}$. Simple algebra shows that $\frac{m(1-m)}{2} > \pi^C \Leftrightarrow m \in (q^C, q_T^C)$. This concludes the proof of Proposition 2.

We have just presented a class of strategies that constitute an equilibrium for $m > q^C$. We have also established that the equilibrium associated with these strategies ensures higher payoff than the Cournot one only if $m \in (q^C, q_T^C)$. It ensues that for $m > q_T^C$ there exists an equilibrium in which both firms produce in-house quantities smaller than the Cournot ones and are worse off than in the Cournot case. We refer to this as a mis-coordination equilibrium.

Proposition 3 (Mis-coordination Equilibrium). *If $m > q_T^C$, then $(R(m), R(m))$ is an equilibrium outcome. This outcome satisfies $(R(m), R(m)) < (q^C, q^C)$ and ensures lower profits than the Cournot one.*

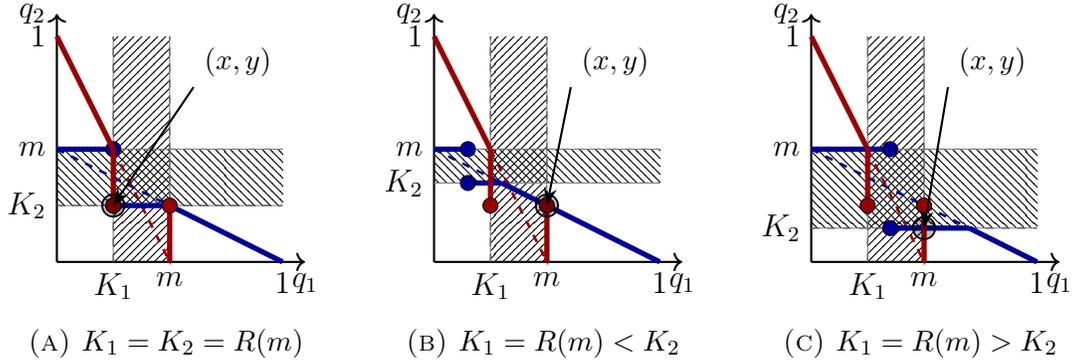


FIGURE 2: CONDITIONAL BEST-RESPONSE FUNCTIONS (BLUE FOR FIRM 1 AND RED FOR FIRM 2) IN DIFFERENT SUBGAMES (K_1, K_2) FOR $m = \frac{1}{2}$

An equilibrium in which both firms produce the Cournot quantity exists for sufficiently large m .

Proposition 4 (Non-Collusive Equilibrium). *If $m \in [q^M, 1]$, then (q^C, q^C) is an equilibrium outcome.*

The proof is constructive and relegated to the appendix. Propositions 1 and 4 together imply that a non-collusive equilibrium exists for $m \notin (q^C, q^M)$. Proposition 6 (see below) establishes that a non-collusive equilibrium exists *only if* $m \notin (q^C, q^M)$.

We characterize next two more equilibria. In these equilibria, one firm outsources and the other produces a smaller quantity in-house. We refer to these asymmetric equilibria as leadership equilibria, to the firm outsourcing as the leader and to the other firm as the follower.

Proposition 5. *If $m > q^C$, then $(m, R(m))$ and $(R(m), m)$ are equilibrium outcomes. These outcomes ensure higher profits than the Cournot one to the firm producing m , and lower profits to the firm producing $R(m)$.*

To sum up, Propositions 2-5 establish that the following equilibria exist:

- If $q^C < m < q^M$: collusive equilibrium and leadership equilibria;
- If $q^M < m < q_T^C$: collusive equilibrium, non-collusive equilibrium and leadership equilibria;
- If $q_T^C \leq m$: mis-coordination equilibrium, non-collusive equilibrium and leadership equilibria.

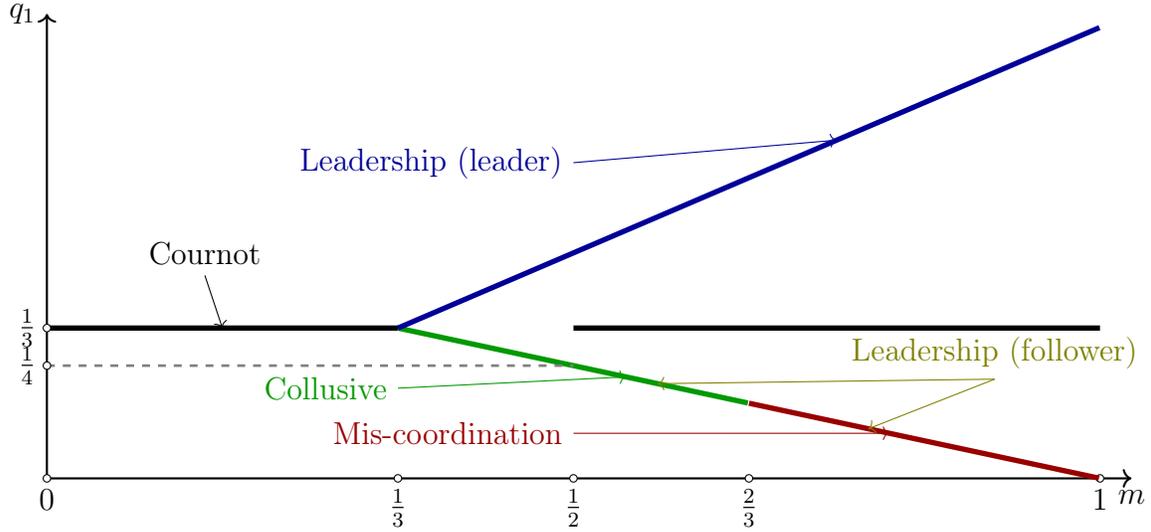


FIGURE 3: EQUILIBRIUM QUANTITIES

Figure 3 and Figure 4 show, respectively, the outcomes and payoffs associated with these equilibria. The last result of the section establishes that the equilibria we have considered so far are all the equilibria of the game.

Proposition 6. *Let $m > q^C$. Propositions 2–5 list all the equilibrium outcomes.*

5 Voluntary Partial Commitment

The duopoly game with capacity and outsourcing can be thought of as a game of *voluntary partial commitment*. Setting capacity K_i is *essentially* equivalent to committing to sell some quantity $q_i \in [0, K_i] \cup [m, 1]$.¹¹ The first-stage choice is then equivalent to picking an element from the set $\tilde{\mathbb{Q}}$ defined as

$$\tilde{\mathbb{Q}} := \{[0, K_i] \cup [m, 1] \mid K_i \in [0, m]\}. \quad (4)$$

We dub this partial commitment, as in the first stage the firm restricts its choice set, but does not fully commit to a specific quantity. Commitment is voluntary because a firm can avoid any commitment by setting capacity as large as the minimum order quantity m .

¹¹Formally, the firm can choose any quantity in the second stage, but quantities in (K_i, m) are never optimal (Lemma 1).

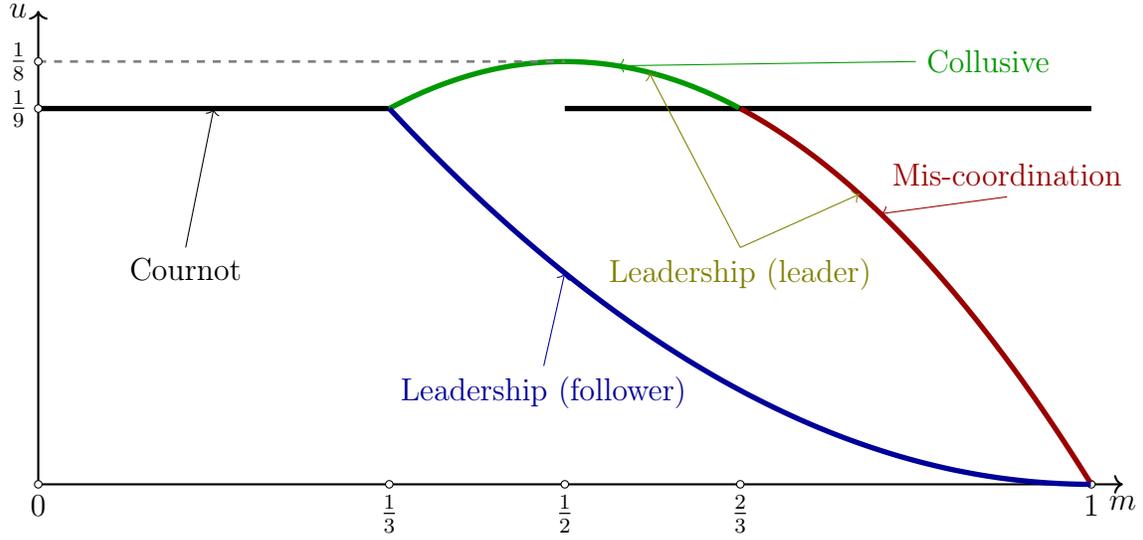


FIGURE 4: EQUILIBRIUM PAYOFFS

In this section, we explore more broadly partial voluntary commitment in the context of the duopoly game. We modify the duopoly game as follows. In the first stage, each firm i picks an element, called a *commitment set*, from a collection of subsets of the interval $[0, 1]$. We focus on sets of subsets that have the entire interval $[0, 1]$ as an element, refer to these sets of subsets as *commitment structures* and denote them \mathbb{Q}_i for each firm i . Firms observe each other's choice of commitment sets and then, in the second period, they select a quantity in their commitment set. The payoff of firm i is $\pi(q_i, q_j)$.

We will use the notation $\Gamma(\mathbb{Q}_1, \mathbb{Q}_2)$ for the game parameterized by the pair of commitment structures $\Gamma(\mathbb{Q}_1, \mathbb{Q}_2)$. For instance, $\tilde{\mathbb{Q}}$ is a commitment structure and the duopoly with capacity and outsourcing is equivalent to the game $\Gamma(\tilde{\mathbb{Q}}, \tilde{\mathbb{Q}})$.

We will say that a pair of quantities (q_1, q_2) is *plausible* if there exists a pair of commitment structures \mathbb{Q}_1 and \mathbb{Q}_2 such that (q_1, q_2) is an equilibrium outcome for the game $\Gamma(q_1, q_2)$.

We characterize here the entire set of plausible outcomes. We already know some plausible outcomes. Propositions 2, 5 and 3 together establish that any outcome such that $q_1 = q_2 \in [0, q^C]$ is plausible. We refer to the set of such outcomes as $S_=$, so

$$S_:= \{(q_1, q_2) \in \mathbb{R}^2 | q_1 = q_2 \in [0, q^C]\}.$$

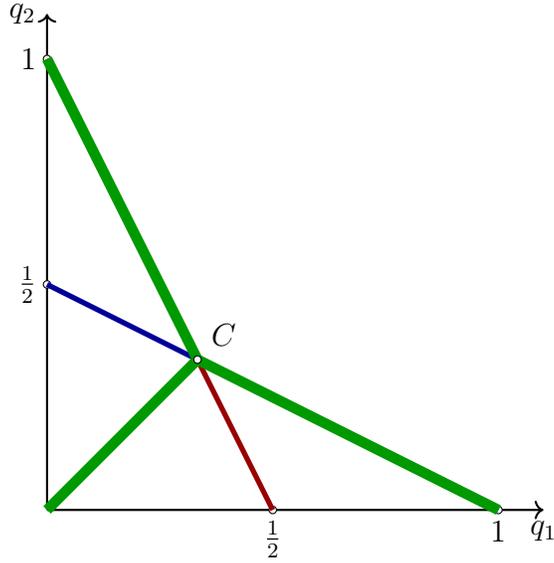


FIGURE 5: THE GREEN LINES SHOW THE SET $S_{=} \cup S_{>}$.

Proposition 5 establishes that any outcome such that $q_i > q^C$ and $q_j = R(q_i)$ is plausible. We refer to the set of such outcomes as $S_{>}$, so that:

$$S_{>} := \{(q_1, q_2) \in \mathbb{R}^2 \mid q_2 \in (q^C, 1], q_1 = R(q_2)\} \cup \{(q_1, q_2) \in \mathbb{R}^2 \mid q_1 \in (q^C, 1], q_2 = R(q_1)\}.$$

Hence, so far we have established that outcomes in $S_{=} \cup S_{>}$ are plausible. We show next that there are no other plausible outcomes.

Figure 5 shows the set $S_{=} \cup S_{>}$.

Proposition 7. *The set of plausible outcomes is $S_{=} \cup S_{>}$.*

Note that $\pi(q, R(q)) = \pi(R(q), R(q))$ for any $q \in [0, 1]$:

$$\pi(q, R(q)) = \frac{(1-q)q}{2} = \pi(R(q), R(q)).$$

We'll refer to this observation as "Property α ".

We prove the proposition by a sequence of lemmata.

Lemma 3. *Let (q_1, q_2) be an equilibrium outcome of the game $\Gamma(\mathbb{Q}_1, \mathbb{Q}_2)$. Then (q_1, q_2) is an equilibrium outcome of game $\Gamma(\mathbb{Q}'_1, \mathbb{Q}'_2)$ where \mathbb{Q}'_i only includes (a) the interval $[0, 1]$, and (b) the subset chosen by firm i in the original equilibrium – as long as the latter is different from $[0, 1]$.*

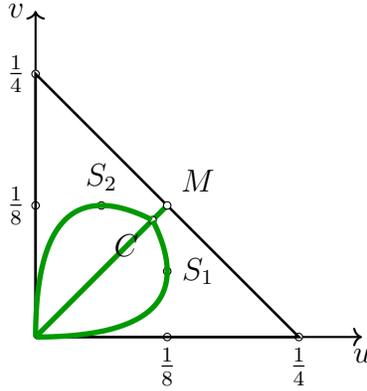


FIGURE 6: THE GREEN AREA DEPICTS ALL PLAUSIBLE PAYOFF PAIRS FROM THIS SECTION

Proof: The argument here is just that \mathbb{Q}'_1 and \mathbb{Q}'_2 can be obtained from \mathbb{Q}_1 and \mathbb{Q}_2 by removing the elements of \mathbb{Q}_1 and \mathbb{Q}_2 that are not selected in the equilibrium associated with outcomes (q_1, q_2) and are different from $[0, 1]$. ■

For the rest of the proof, we focus, without further mentioning it, on \mathbb{Q}_1 and \mathbb{Q}_2 such that (a) $|\mathbb{Q}_1| = 2$ and $|\mathbb{Q}_2| = 2$ and (b) $[0, 1] \in \mathbb{Q}_1 \cap \mathbb{Q}_2$.

Lemma 4. *In any equilibrium of the game $\Gamma(\mathbb{Q}_1, \mathbb{Q}_2)$ in which in stage 1 at least one firm selects $[0, 1]$, the equilibrium outcome $(q_1, q_2) \in S_{>}$.*

Proof: Consider an equilibrium in which firm 1 selects $[0, 1]$ in stage 1 (the argument in case firm 2 selects $[0, 1]$ in stage 1 is identical and omitted). The equilibrium outcome (q_1, q_2) must be such that (a) $q_1 = R(q_2)$, (obvious) and (b) $\pi(q_2, q_1) \geq \frac{1}{9}$ (firm 2 can ensure utility $\frac{1}{9}$ by selecting $[0, 1]$ in stage 1). These two conditions are satisfied at the same time only if $(q_1, q_2) \in S_{>}$. ■

Lemma 5. *In any equilibrium of the game $\Gamma(\mathbb{Q}_1, \mathbb{Q}_2)$, in which in stage 1 neither firm selects $[0, 1]$:*

1. *if the equilibrium outcome (q_1, q_2) is such that $\max\{q_1, q_2\} > q^C$, then $(q_1, q_2) \in S_{>}$,*
2. *if the equilibrium outcome (q_1, q_2) is such that $\max\{q_1, q_2\} \leq q^C$, then $(q_1, q_2) \in S_{=}$.*

Proof: Let (q_1, q_2) denote the outcome of an equilibrium of $\Gamma(\mathbb{Q}_1, \mathbb{Q}_2)$. Let the equilibrium be such that neither firm selects $[0, 1]$ in stage 1.

1. Suppose that $q_i > q^C$. We argue that $q_j = R(q_i)$. We proceed by contradiction. Suppose $q_j \neq R(q_i)$. Firm j can do better by deviating to selecting $[0, 1]$ in the first stage. To see this note that, as $q_i \geq q^C$, then it is easy to verify that $\pi(q_i, R(q')) > \pi(q', R(q'))$ for all $q' > q_i$. Hence, if firm j deviates to select $[0, 1]$ in the first stage, then firm i will select some quantity not larger than q_i and firm j will best respond. As on path firm i selects q_i and firm 2 selects $q_j \neq R(q_i)$, the deviation in stage 1 is profitable for firm j .

This contradiction ensures that $q_j = R(q_i)$, and therefore $(q_1, q_2) \in S_{>}$.

2. Suppose that $q_1 < q^C$, and $q_2 < q^C$. Let C_1 and C_2 denote, respectively, the elements of \mathbb{Q}_1 and \mathbb{Q}_2 that are not $[0, 1]$. Then:

- (a) as (q_1, q_2) is an equilibrium of the subgame (C_1, C_2) , then for any quantity $\tilde{q} \in C_i$,

$$\pi(q_i, q_j) \geq \pi(\tilde{q}, q_j), \quad (5)$$

It must thus be the case that $C_1 \cap (q_1, q^C] = \emptyset$. By the same argument, $C_2 \cap (q_2, q^C] = \emptyset$.

- (b) After firm j deviates to $[0, 1]$ in the first stage, the equilibrium in the associated subgame must be such that firm i chooses some quantity \bar{q}_i and firm j chooses $R(\bar{q}_i)$. Clearly $\bar{q}_i > q_i$, else firm j would gain from deviating to $[0, 1]$ at stage 1. As $C_i \cap (q_i, q^C] = \emptyset$, then $\bar{q}_i > q_i$ implies $\bar{q}_i > q^C$. Moreover, as $q_i \in C_i$, a necessary condition for the subgame following the deviation of firm j being such that firm i chooses some quantity \bar{q}_i and firm j chooses $R(\bar{q}_i)$ is:

$$\pi(\bar{q}_i, R(\bar{q}_i)) \geq \pi(q_i, R(\bar{q}_i)). \quad (6)$$

Adding up (5) and (6) gives us

$$(\pi(\bar{q}_i, R(\bar{q}_i)) - \pi(\bar{q}_i, q_j)) - (\pi(q_i, R(\bar{q}_i)) - \pi(q_i, q_j)) \geq 0$$

Simplifying the left side we get:

$$(q_j - R(\bar{q}_i))(\bar{q}_i - q_i) \geq 0.$$

As $\bar{q}_i > q_i$, we get that $q_j \geq R(\bar{q}_i)$. By a symmetric argument $q_i \geq R(\bar{q}_j)$.

Note also that

$$\pi(R(\bar{q}_i), R(\bar{q}_i)) = \pi(\bar{q}_i, R(\bar{q}_i)) \geq \pi(q_i, R(\bar{q}_i)).$$

where the equality follows from Property α and the inequality from (6). As $q_i < q^C < \bar{q}_i$ and $u(\cdot, R(\bar{q}_i))$ is a strictly concave function, this is only possible if $q_i \leq R(\bar{q}_i)$. By a symmetric argument, $q_j \leq R(\bar{q}_j)$.

Combining these observations, we get $R(\bar{q}_i) \leq q_i \leq R(\bar{q}_j)$ and $R(\bar{q}_j) \leq q_j \leq R(\bar{q}_i)$, which means $R(\bar{q}_i) = R(\bar{q}_j)$, and therefore $q_i = q_j$. This proves that $(q_1, q_2) \in S_+$.

■

Lemmata 4 and 5 together imply that in any equilibrium of the game $\Gamma(\mathbb{Q}_1, \mathbb{Q}_2)$ the equilibrium outcome $(q_1, q_2) \in S$. This concludes the proof of the proposition.

6 Conclusions

In this paper we show that tacit collusion need not rely on repeated interaction or patience: it can emerge in a one-shot Cournot duopoly once firms can choose capacity and, if needed, outsource subject to minimum order quantities. The MOQ creates a gap in feasible outputs that makes “produce a lot” a credible punishment following a deviation in capacity, thereby sustaining equilibria in which both firms set capacity below the Cournot level and earn higher profits for an intermediate range of MOQs. Alongside this collusive equilibrium, the model also admits leadership equilibria and (when the MOQ is large) a miscoordination equilibrium, and we fully characterize the equilibrium outcomes across parameter values. Finally, by recasting the environment as a game of voluntary partial commitment, we clarify which outcomes can be rationalized by commitment to restricted action sets and highlight how limited commitment alone can generate collusive-looking market outcomes even in static settings.

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Appendix - Proofs

Proof of Lemma 1: Suppose by contradiction that firm i chooses $q_i \in (K_i, m)$. If the other firm chooses some quantity q_j , then firm i 's profits are $\tilde{\pi}(q_i, q_j)$, given by:

$$\tilde{\pi}(q_i, q_j) = (1 + c - q_i - q_j)q_i - cm,$$

which has partial derivative $\tilde{\pi}_1 = 1 + c - 2q_i - q_j$. Note that in equilibrium $q_i + q_j < 1$, otherwise at least one firm would earn strictly negative profit. Hence:

$$\tilde{\pi}_1 = 1 - q_1 - q_2 + c - q_1 > c - q_1 > c - m,$$

where the first inequality follows from $q_i + q_j < 1$ and the second inequality from $q_i < m$. Assumption 1 then guarantees that firm i would be better off choosing a marginally higher quantity. This contradiction implies that in all subgame equilibria firm i picks $q_i \notin (K_i, m)$. ■

Proof of Proposition 1: Let $m \leq q^C$. We first show that firm $i \in \{1, 2\}$ can ensure a payoff at least as large as π^C by choosing $K_i = m$.

Consider the subgame associated with $K_i = m$ and some K_j . In any subgame equilibrium, $q_i = R(q_j)$ and $q_j \leq q^C$. The first property is immediate. The second can be verified as follows. As $q > R(R(q)) > q^C$ for any $q > q^C$, then $q_j > q^C$ cannot be part of a subgame equilibrium, as in such equilibrium $q_i = R(q_j)$ and firm j would have a profitable deviation. Hence – by choosing $K_i = m$ – firm i , ensures a payoff of $\pi(R(q_j), q_j)$ for some $q_j \leq q^C$. Note that:

$$\pi(R(q_j), q_j) > \pi(q^C, q_j) \geq \pi^C.$$

Hence, in equilibrium both firms earn at least π^C .

Suppose in equilibrium $q_i < q^C$. As $R(q_i) > q^C \geq m$, then in equilibrium $q_j = R(q_i)$. Yet $q_i < q^C \Rightarrow \pi(q_i, R(q_i)) < \pi^C$, which contradicts our previous remark. Hence, in equilibrium $q_i \geq q^C$. Let $q_i > q^C$ and $q_j \geq q^C$; then $\pi(q^C, q_j) > \pi(q_i, q_j)$. As $q_i = q^C$ is a feasible and profitable deviation, then we can rule out $q_i > q^C$. Hence $q_i = q^C$. As $R(q^C) = q^C$, then $q_i = q^C \Rightarrow q_j = q^C$. Hence, in equilibrium $q_i = q^C$ for $i \in \{1, 2\}$. ■

Proof of Proposition 4: Consider strategies such that, for $i \in \{1, 2\}$: $K_i = m$ and

- following this choices of capacity - $q_i = q^C$. These strategies ensure – on path – a subgame equilibrium, so it is enough to look for deviations at the first stage.

If firm i deviates to $K_i \in [q^C, m)$, then $q_i = q_j = q^C$ is a subgame equilibrium outcome. This equilibrium makes the deviation unprofitable.

Consider deviations to $K_i < q^C$. Let the subgame equilibrium in this case be $q_i = K_i$ and $q_j = R(K_i)$. Firm j selects the unconstrained best response. For firm i , note that

$$K_i < q^C \Rightarrow R(R(K_i)) \in (K_i, q^C),$$

and

$$(K_i, q^C) \subset (K_i, x^M) \subseteq (K_i, m).$$

By concavity of the best response function,

$$R(R(K_i)) \in (K_i, m) \Rightarrow R_{K_i}(R(K_i)) \in \{K_i, m\}.$$

Moreover, as $m \geq q^M$ and $R(R(K_i)) \in (K_i, q^M)$, then $u(q^M, R(q_i)) \geq u(m, R(q_i))$. Hence, to verify that $u(q_i, R(q_i)) \geq u(m, R(q_i))$ it is sufficient to verify that $u(q_i, R(q_i)) \geq u(q^M, R(q_i))$ for any $q_i \in [0, q^C]$. This condition holds as:

$$\begin{aligned} \pi(q_i, R(q_i)) \geq \pi(q^M, R(q_i)) &\Leftrightarrow \\ (1 - q_i - \frac{1 - q_i}{2})x &\geq (1 - \frac{1}{2} - \frac{1 - q_i}{2})\frac{1}{2} \Leftrightarrow \\ q_i - 2q_i^2 &\geq 0 \Leftrightarrow q_i \in \left[0, \frac{1}{2}\right] = [0, q^M]. \end{aligned}$$

So $q_i = K_i$, and $q_j = R(K_i)$ is a subgame equilibrium. For this equilibrium, the deviation is not profitable. ■

Proof of Proposition 5: We show here that $q_i = m$ and $q_j = R(m)$ is an equilibrium outcome for $i, j \in \{1, 2\}$. For the subgame associated with $K_i = 0$ and $K_j = R(m)$, quantities $q_i = m$ and $q_j = R(m)$ are a subgame equilibrium outcome (see (2)).

Consider first-stage deviations of firm i . If firm i deviates to $K_i \geq R(m)$, then $q_i = R(m)$ and $q_j = m$ is a subgame equilibrium. If instead $K_i < R(m)$, then $q_i = m$ and $q_j = R(m)$ is a subgame equilibrium. So for every deviation of firm i there is a subgame equilibrium for which the deviation is not profitable.

Consider first-stage deviations of firm j . If firm j deviates to $K_j > R(m)$, quantities

$q_i = m$ and $q_j = R(m)$ are a continuation equilibrium, and this equilibrium makes the deviation unprofitable. Suppose instead that $K_j < R(m)$. If $m \geq \frac{1}{2}$, then $q_i = m$ and $q_j = K_j$ is a continuation equilibrium, and this continuation equilibrium makes the deviation unprofitable. Suppose instead $m < \frac{1}{2}$. We distinguish 2 cases. If $\pi(m, m) \geq \pi(K_j, m)$, then $q_i = m, q_j = m$ is a continuation equilibrium. If instead $\pi(m, m) < \pi(K_j, m)$, then $q_i = m, q_j = K_j$ is a continuation equilibrium. In either case, the deviation is not profitable.

We have thus established that $q_i = m, q_j = R(m)$ is an equilibrium outcome. ■

To prove Proposition 6 we rely on a sequence of Lemmas.

Lemma 6. *For any subgame (K_1, K_2) , any subgame equilibrium outcome such that $q_1 \notin \{K_1, m\}$ and $q_2 \notin \{K_2, m\}$ corresponds to the Cournot outcome.*

Proof: Note that $\pi(q_i, q_j)$ is a differentiable function of both arguments, and $\frac{d\pi(q_i, q_j)}{dq_i} = 0$ if and only if $q_i = R(q_j)$. Hence for a subgame (K_1, K_2) , in every subgame equilibrium, $q_1 \in \{K_1, m, R(q_2)\}$ and $q_2 \in \{K_2, m, R(q_1)\}$. Hence $q_1 \notin \{K_1, m\}$ and $q_2 \notin \{K_2, m\}$ imply, respectively, $q_1 = R(q_2)$ and $q_2 = R(q_1)$. These two conditions can hold at the same time only if $q_1 = q_2 = q^C$. ■

The next lemma sets a lower bound to the payoff that a firm gets in equilibrium.

Lemma 7. *Let $m > q^C$. In equilibrium, both firms gets a payoff not smaller than $\pi(R(m), m)$.*

Proof: In any subgame following $K_i = m$, every continuation equilibrium is such that, (i) $q_i = R(q_j)$, and (ii) $q_j \leq m$. Observation (i) is immediate. Observation (ii) follows from Lemma 6, as $R(q) < K_i$ for any $q > m$.

Note that $\pi(R(q), q) = (\frac{1-q}{2})^2$ is a decreasing function of q . Hence, in light of observations (a) and (b), by selecting $K_i = m$ firm i ensures a payoff not smaller than $\pi(R(m), m)$. The lemma follows. ■

The next two lemmas are immediate corollaries of Lemma 7.

Lemma 8. *Let $m > q^C$. Any equilibrium outcome is such that $\max\{q_1, q_2\} \leq m$.*

Proof: For any $q_j > m$ and any q_i :

$$\pi(q_i, q_j) \leq \pi(R(q_j), q_j) < \pi(R(m), m),$$

where the first inequality follows from the definition of function R , and the second holds as $\pi(R(q), q)$ is a decreasing function of q . The highlighted inequalities, together with Lemma 7, imply that there is no equilibrium outcome such that some firm i chooses $q_i > m$, as for such an outcome the competitor would have a profitable deviation in the first stage, which is a contradiction. ■

Lemma 9. *Let $m > q^C$. In any equilibrium outcome, if firm i picks $q_i = m$, then the other firm select $q_j = R(m)$.*

Proof: From the definition of function R , we know that $\pi(q, m) < \pi(R(m), m)$ for any $q \neq R(m)$ (call this Observation 1). We conclude that there is no equilibrium outcome such that firm i selects $q_i = m$ and the other firm selects $q_j \neq R(m)$, as Observation 1 and Lemma 7 together imply that firm j would have a profitable deviation in the first stage. ■

Next we rule out equilibrium outcomes in which some firm i picks $q_i \in (q^C, m)$.

Lemma 10. *Let $m > q^C$. An equilibrium outcome such that some firm i picks $q_i \in (q^C, m)$ does not exist.*

Proof: Suppose an equilibrium outcome with $q_1 \in (q^C, m)$ exist. Without loss of generality we can assume $q_1 \geq q_2$ (for $q_1 < q_2$ Lemmas 8 and 9 imply that $q_2 < m$, hence the same proof holds, up to a relabeling).

On path firm 1 selects $K_1 \in [q_1, m]$. So $q_1 \leq R(q_2)$, else firm 1 would have a profitable deviation. As $q_1 > q^C$ then $q_1 > R(R(q_1))$. Hence $R(R(q_1)) < R(q_2)$, which is equivalent to $R(q_1) > q_2$. This last inequality implies $K_2 = q_2 < m$, else firm 2 would have a profitable deviation in the second stage.

Consider a deviation by firm 2 in the first stage to $K_2 = m$. Following this deviation, in every subgame equilibrium $q_2 = R(q_1)$; hence there is no subgame equilibrium such that $q_1 = m$, as this would imply $q_2 = R(m)$, and, by property α , as $K_1 > R(m)$, then firm 1 would have a profitable deviation to some quantity $q \in (R(m), K_1)$. So, in any subgame equilibrium following the deviation, firm 1 selects some $q_1 \leq K_1$. Any equilibrium must then be such that $q_1 \leq K_1$ and $q_2 \leq K_2$. As, moreover, $K_1 > q^C$ and $K_2 > q^C$, by Lemma 6, the only subgame equilibrium following the deviation is the Cournot one. The deviation is profitable for firm 2 as:

$$\pi(q^C, q^C) = \pi(R(q^C), q^C) > \pi(R(q_1), q_1) \geq \pi(q_2, q_1),$$

where the first inequality holds as $q'_i > q^C$ and $\pi(R(q), q)$ is a decreasing function of q . This contradiction proves the lemma. \blacksquare

We first establish that Cournot is not an equilibrium outcome for $m \in (q^C, q^M)$.

Lemma 11. *Let $m \in (q^C, q^M)$. The Cournot outcome is not an equilibrium outcome.*

Proof: We proceed by contradiction. Suppose an equilibrium exists associated with the Cournot equilibrium outcome. In such equilibrium, the firms select $K_1, K_2 \geq q^C$ in stage 1 and earn π^C . Yet firm i can earn more by deviating to $K_i = 0$. To see this, note that $q_i = m$ and $q_j = R(m)$ is a continuation equilibrium of the the subgame associated the deviation. Moreover this is the only continuation equilibrium, as (a) a continuation equilibrium with $q_i > m$ does not exist by Lemma 6 (to see this note that $R(q) < K_j$ for all $q > m$) and (b) an equilibrium with $q_i = 0$ does not exist as in such equilibrium we would have $q_j = R(q_i) = q^M$, yet for $q_j = q^M$ firm i has profitable deviation as $\pi(m, q^M) > \pi(0, q^M)$. Finally, note that $\pi(m, R(m)) > \pi^C$ for $m \in (q^C, q^M)$. \blacksquare

Lemma 12. *Let $m > q^C$. An equilibrium such that $\max\{q_1, q_2\} \leq q^C$ must be such that either $q_1 = q_2 \in \{R(m), q^C\}$.*

Proof: Consider an equilibrium in which firm i selects $q_i = q^C$ and the other selects $q_j < q^C$. Clearly in the first stage the firms select, respectively, some $K_i \geq q^C$ and $K_j = q_j$. But then, firm j could guaranteed the Cournot outcome (q^C, q^C) by deviating to $K_j \geq q^C$. This contradiction proves that if $q_i = q^C$ then $q_j^* = q^C$.

Consider now an equilibrium in which firms select quantities such that $q_i \leq q_j < q^C$. Note first that $q_j < q^C$ and $q_i \leq q_j$ are a subgame equilibrium only if the subgame is such that $K_i = q_i$ and $K_j = q_j$.

We proceed now by contradiction. Suppose first that $K_j < R(m)$, and therefore $K_i = q_i \leq q_j = K_j < R(m)$. Then q_i and q_j cannot be a subgame equilibrium, as property α ensures that firm i would gain from deviating to $q_i = m$ in the last stage.

Suppose that $K_j > R(m)$. Then firm i has a profitable deviation to $K_i = m$. To see this, note that the subgame following $K_i = m$ and $K_j < q^C$ has only equilibria such that $q_j \leq K_j$ and $q_i = R(q_j)$. To see this it is enough to note that if $q_j = m$ and there fore $q_i = R(m)$, then by property α and $K_j > R(m)$ firm 2 would have a profitable deviation. Any such continuation equilibrium makes the deviation profitable for firm i .

We conclude that $K_j = R(m)$. Suppose $K_i < R(m)$. Then in the last stage firm j would have a profitable deviation to $q_j = m$, by property α . This contradiction establishes that $K_i = K_j = R(m)$ and concludes the proof. ■