Identification in Structural Vector Autoregressive models with structural changes, with an application to U.S. monetary policy∗

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Abstract

A growing line of research makes use of structural changes and different volatility regimes found in the data in a constructive manner to improve the identification of structural parameters in Structural Vector Autoregressions (SVARs). A standard assumption made in the literature is that the reduced form unconditional error covariance matrix varies while the structural parameters remain constant. Under this hypothesis, it is possible to identify the SVAR without needing to resort to theory-driven restrictions. With macroeconomic data, the assumption that the transmission mechanism of the shocks does not vary across volatility regimes is debatable. We derive novel necessary and sufficient rank conditions for local identification of SVARs, where both the error covariance matrix and the structural parameters are allowed to change across volatility regimes. Our approach generalizes the existing literature on ‘identification through changes in volatility’ to a broader framework and opens up interesting possibilities for practitioners. An empirical illustration focuses on a small monetary policy SVAR of the U.S. economy and suggests that monetary policy has become more effective at stabilizing the economy since the 1980s.

Keywords: Heteroskedasticity, Identification, Monetary policy, Structural VAR.
J.E.L. C32, C50, E52.

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1 Introduction

Structural Vector Autoregressions (SVAR) are widely used for policy analysis and to provide stylized facts about business cycle. As is known, it is necessary to identify the structural shocks to run policy simulations. The publication of Rigobon’s (2003) seminal paper seems to have suggested that structural changes - in particular, changes in the reduced form unconditional covariance matrix - can be used constructively to improve the identification of structural parameters that are assumed to be stable over time and across volatility regimes.\(^1\) In this setup, the information that there exist different volatility regimes in the data represents an ‘additional’ identification source that can be exploited to identify the shocks without the need to resort to other type of identification restrictions. Lanne and Lütkepohl (2008) have extended and formalized this idea to the case of SVARs, see also Lanne and Lütkepohl (2010), Lanne et al. (2010) and Ehrmann et al. (2011).\(^2\)

The hypothesis that the data generating process is a VAR with constant parameters apart from changes in the volatility of the disturbances appears reasonable in certain applications, but is in general questionable with macroeconomic data. It is well recognized, indeed, that structural breaks may have marked consequences on both the transmission and propagation mechanisms of the shocks. Existing macroeconomic evidence suggests that there is no compelling reason to believe that the occurrence of structural breaks that change the variance of the data have no impact on the transmission mechanism of the shocks.

The main motivation of the present paper is that ‘the identification approach via changes in volatility’ suggested by Rigobon (2003) and extended by Lanne and Lütkepohl (2008) to the case of SVARs, can be generalized to a broader framework, opening up interesting possibilities for practitioners. By applying the seminal identification rules of Rothenberg (1971), we derive novel necessary and sufficient rank conditions for local identification which apply when discrete permanent (not recurring) breaks occur simultaneously in the reduced form VAR error covariance matrix and in the (structural) parameters which define the relationships between the VAR disturbances and the structural shocks. The results in Rigobon (2003) and Lanne and Lütkepohl (2008) obtain as special cases of our analysis. Unlike Rigobon (2003) and Lanne and Lütkepohl (2008), in our setup the patterns of SVAR impulse response functions may change across volatility regimes.

As is known, structural changes offer identifying power only if some parameters do not change. The difficult open question is what these parameters are. In our approach, different structural models are imposed on different volatility regimes through a balanced combination.

\(^1\)In the recent literature, Sentana (1992) and Sentana and Fiorentini (2001) have introduced similar ideas in the context of factor models, Klein and Vella (2010) and Lewbel (2010) in the context of simultaneous systems of equations and Mavroeidis and Magnusson (2010) in the context of dynamic models featuring forward-looking behavior. See also Keating (2004) for the case of SVARs.

\(^2\)Other examples include Caporale et al. (2005a), Dungey and Martin (2001), King et al. (1994), Caporale et al. (2005b), Rigobon and Sack (2003, 2004) and Normandin and Phaneuf (2004).
of the statistical information provided by the data and ‘conventional’ linear restrictions. It is economic reasoning that provides indications about which are the structural parameters likely to change across the volatility regimes, and which are the structural parameters which are likely to remain unchanged.

Our approach opens up interesting possibilities for practitioners. We discuss new identification strategies that stem from our identification conditions, using examples taken from the empirical monetary policy literature. SVARs which would typically be unidentified in the case of constant parameters, can be identified (overidentified) when the structural parameters are allowed to vary across volatility regimes. Moreover, when the restrictions are overidentifying, the practitioner can monitor the data adequacy of his/her specified SVAR by computing likelihood-based tests.

To illustrate the usefulness of our approach, we identify and estimate a small monetary policy SVAR for the U.S. economy, using quarterly data and two regimes of volatility. Our empirical evidence suggests that in spite of a general reduction in the volatility of the shocks, the response of U.S. monetary policy changed in the move from the ‘pre-Volcker’ period to the ‘Great Moderation’ period.

The remainder of this paper is organized as follows. In Section 2 we discuss the backgrounds and motivations of our paper. In Section 3 we derive our main result by providing the necessary and sufficient rank conditions for local identification. In Section 4 we present an empirical illustration where we estimate a small monetary policy SVAR of the U.S. economy, using quarterly data and two volatility regimes. Section 5 contains some concluding remarks. A Technical Supplement complements the results of the paper in several dimensions.\footnote{The Technical Supplement is available at http://www.rimini.unibo.it/fanelli/TS_Bacchiocchi_Fanelli.pdf}

Throughout the paper we use the following notation, matrices and conventions, most of which are taken from Magnus and Neudecker (2007). $K_n$ is the $n^2 \times n^2$ commutation matrix, i.e. the matrix such that $K_n \text{vec}(M) = \text{vec}(M')$ where $M$ is $n \times n$, and $D_n$ is the duplication matrix, i.e. the $n^2 \times \frac{1}{2}n(n+1)$ full column rank matrix such that $D_n \text{vech}(M) = \text{vec}(M)$, where $\text{vech}(M)$ is the column obtained from $\text{vec}(M)$ by eliminating all supra-diagonal elements. Given $K_n$ and $D_n$, $N_n := \frac{1}{2}(I_{n^2} + K_n)$ is a $n^2 \times n^2$ matrix such that $\text{rank}[N_n] = \frac{1}{2}n(n+1)$ and $D_n^+ := (D_n' D_n)^{-1} D_n'$ is the Moore-Penrose inverse of $D_n$. Given a $n \times n$ diagonal matrix $M$, $w(M)$ is the $n \times 1$ vector $w(M) := (m_{11}, \ldots, m_{nn})'$, and we denote with $U_n$ the $n \times n^2$ full row rank matrix with the property that $U_n' w(M) = \text{vec}(M)$. Finally, when we say that the matrix $M := M(v)$, whose elements depend (possibly nonlinearly) on the elements of the vector $v$, ‘has rank $r$ evaluated at $v_0$’, we mean that $v_0$ is a ‘regular point’, i.e. that $\text{rank}(M)=r$ does not change within a neighborhood of $v_0$.\footnote{The Technical Supplement is available at http://www.rimini.unibo.it/fanelli/TS_Bacchiocchi_Fanelli.pdf}
2 Background and motivations

In this section, we present the basic econometric framework upon which our analysis will be developed. To fix main ideas and notation, we first review the standard approach to the identification of SVARs (Sub-section 2.1), and then move to the mechanics of the ‘identification via changes in volatility’ method (Sub-section 2.2). Finally, we sketch our contribution in this literature (Sub-section 2.3).

2.1 Standard identification approach

Let $Z_t$ be the $n \times 1$ vector of observable variables. Our reference reduced form model is given by the VAR system with constant parameters:

$$Z_t = A_1 Z_{t-1} + \ldots + A_k Z_{t-k} + \Psi D_t + \varepsilon_t , \quad t = 1, \ldots, T \tag{1}$$

where $\varepsilon_t$ is a $n$-dimensional White Noise process with positive definite time-invariant covariance matrix $\Sigma_\varepsilon := E(\varepsilon_t \varepsilon_t')$, $A_j$, $j = 1, \ldots, k$ are $n \times n$ matrices of time-invariant coefficients, $k$ is the VAR lag order, $D_t$ is an $m$-dimensional vector containing deterministic components (constant, trend and dummies), and $\Psi$ is the $n \times m$ matrix of associated coefficients. $T$ is the sample length.

We compact the VAR system (1) in the expression

$$Z_t = \Pi W_t + \varepsilon_t , \quad t = 1, \ldots, T \tag{2}$$

where $W_t := (Z'_{t-1}, \ldots, Z'_{t-k}, D'_t)$ and $\Pi := (A, \Psi)$. The matrix $\Pi$ is $n \times f$, $f := \text{dim}(W_t) := nk + m$, and the VAR reduced form parameters are collected in the $p$-dimensional vector $\theta := (\pi', \sigma^+)'$, where $\pi := \text{vec}(\Pi)$ and $\sigma^+ := \text{vech}(\Sigma)$, $p := nf + \frac{1}{2}n(n + 1)$.

The SVAR we are interested in this paper is defined by

$$\varepsilon_t := C e_t , \quad E(e_t e_t') := I_n , \quad \Sigma_\varepsilon = CC' \tag{3}$$

where $C$ is a non-singular $n \times n$ matrix of structural parameters and $e_t$ is a $n$-dimensional i.i.d. vector of structural shocks with covariance matrix normalized to $I_n$. As is known, the system (2)-(3) is unidentified without any restriction on the elements of the $C$ matrix. The standard way to achieve identification is to include a set of linear restrictions on $C$ that we write in

\[\text{We consider the formulation in Eq. (3) of the SVAR (the ‘C-model’, using the terminology in Amisano and Giannini, 1997) because it is largely used in empirical analysis, although our approach is consistent with the alternative specification }\]

$$K \varepsilon_t := e_t , \quad E(e_t e_t') := I_n$$

(termed ‘K-model’ in Amisano and Giannini, 1997) where $K := C^{-1}$.\footnote{We consider the formulation in Eq. (3) of the SVAR (the ‘C-model’, using the terminology in Amisano and Giannini, 1997) because it is largely used in empirical analysis, although our approach is consistent with the alternative specification }
In Eq. (4), $S_C$ is a $n^2 \times a_C$ selection matrix, $\gamma$ is $a_C \times 1$ and contains the ‘free’ elements of $C$, and $s_C$ is a $n^2 \times 1$ vector. The information required to specify the matrix $S_C$ and the vector $s_C$ usually comes from the economic theory or from structural and institutional knowledge related to the problem under study. The condition $a_C := \dim(\gamma) \leq n(n+1)/2$ is necessary for identification. Necessary and sufficient condition for identification is that the $n(n+1)/2 \times a_C$ matrix

$$2D_n^+ (C \otimes I_n) S_C$$

has full column rank evaluated at $C_0$, where $C_0$ denotes the counterpart of $C$ that fulfills the restriction $\text{vec}(C_0) := S_C \gamma_0 + s_C$, and $\gamma_0$ is the ‘true’ value of $\gamma$, see e.g., Giannini (1992), Hamilton (1994) and Amisano and Giannini (1997). To avoid confusion, throughout the paper we call ‘reduced form parameters’ the elements in the vector $\theta$, and ‘structural parameters’ the elements of the vector $\gamma$ and, possibly, the variances of the structural shocks $\epsilon_t$ when these are not normalized to one. If the rank condition in Eq. (5) holds, the orthogonalized impulse response functions (IRFs) are taken from the matrices

$$\Xi_h := [\psi_{lm,h}] := \Phi_h \hat{C} := (J' \hat{A}^h J) \hat{C}, \quad h = 0, 1, 2, \ldots,$$

where

$$\hat{A} := \begin{pmatrix} A \\ I_{n(k-1)} \end{pmatrix} \in \mathbb{R}^{n(k-1) \times n}$$

is the VAR companion matrix, $J := (I_n, 0, \ldots, 0)$ and $\hat{C}$ denotes a specification of $C$ such that the matrix in Eq. (5) has full column rank. The coefficient $\psi_{lm,h}$ captures the response of variable $l$ to a one-time impulse in variable $m$, $h$ periods before.

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5Giannini (1992) and Amisano and Giannini (1997) derive the necessary and sufficient identification rank condition by referring to linear restrictions in ‘implicit form’. The necessary and sufficient rank condition in Eq. (5) can be checked ex-post at the ML estimate but also prior to estimation at random points drawn uniformly from the parameter space, see e.g. Giannini (1992). Iskrev (2010) applies the same idea to check the identification of DSGE models. Lucchetti (2006), instead, has shown that Eq. (5) can be replaced with a ‘structure condition’ which is independent on the knowledge of the structural parameters but is still confined to the local identification case. Rubio-Ramirez et al. (2010) have established novel sufficient conditions for global identification in SVARs and necessary and sufficient conditions for exactly identified systems. In this paper, we confine our attention to the general case of possibly over-identified SVARs and to necessary, other than sufficient identification conditions. Indeed, we deem important to have a criterion to establish whether a given parameter point selected from the admissible structural parameter space does not identify the SVAR.

6The identification of $C$ can also be achieved by complementing the symmetry restrictions $\Sigma = CC'$ with a proper set of constraints on the matrix

$$\Xi_\infty := (I_n - A_1 - \cdots - A_k)^{-1}C := \sum_{h=0}^{\infty} \Phi_h C := J'(I_{nk} - \hat{A})^{-1}JC$$

which measures the long run impact of the structural shocks on the variables (Blanchard and Quah, 1989). Constraints on $\Xi_\infty$ can be used in place of, or in conjunction with, the ‘short run’ restrictions in Eq. (4).
2.2 Identification through heteroskedasticity

In a seminal contribution, Rigobon (2003) proposed an alternative way to solve the identification problem in simultaneous systems of equations that extends to the case of SVARs. The distinctive feature of Rigobon’s (2003) approach is that when the data are characterized by (at least) two different regimes of volatility, the identification of the shocks can be achieved without linear constraints of the type in Eq. (4).

Without any loss of generality, we consider a bivariate SVAR model for the vector $Z_t = (Z_{1t}, Z_{2t})'$ and assume that the data generating process is given by the system (2)-(3). We further assume that at time $t=T_B$, where $1 < T_B < T$, the variance of the data changes in the sense that the two sets of observations $Z_1, ..., Z_{T_B}$ and $Z_{T_B+1}, ..., Z_T$ are characterized by the two VAR covariance matrices $\Sigma_{\varepsilon, 1}$ and $\Sigma_{\varepsilon, 2}$, respectively, where

$$\Sigma_{\varepsilon, i} := \begin{pmatrix} \sigma_{11,i} & \sigma_{12,i} \\ \sigma_{21,i} & \sigma_{22,i} \end{pmatrix}, \quad i = 1, 2.$$ 

Consider the relationship $\varepsilon_t := C \epsilon_t$, where $\epsilon_t$ is the vector of structural shocks. Rigobon’s (2003) identification approach is based on the joint use of the moment conditions

$$\Sigma_{\varepsilon, 1} = C\Lambda_1 C', \quad \Sigma_{\varepsilon, 1} = C\Lambda_2 C'$$

(7)

where $C$, $\Lambda_1$ and $\Lambda_2$ are defined as

$$C := \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad \Lambda_1 := \begin{pmatrix} \lambda_{11,1} & 0 \\ 0 & \lambda_{22,1} \end{pmatrix}, \quad \Lambda_2 := \begin{pmatrix} \lambda_{11,2} & 0 \\ 0 & \lambda_{22,2} \end{pmatrix}$$

and the matrices $\Lambda_1$ and $\Lambda_2$ collect the variances of the structural shocks in the two volatility regimes. Eq. (7) links the reduced form coefficients $\sigma_+ := (vech(\Sigma_{\varepsilon, 1}'), vech(\Sigma_{\varepsilon, 1}'))' = (\sigma_{11,1}, \sigma_{22,1}, \sigma_{21,1}, \sigma_{11,2}, \sigma_{22,2}, \sigma_{21,2})'$ to the structural form parameters in the matrices $C$, $\Lambda_1$ and $\Lambda_2$. If, as is standard in the SVAR literature, $\Lambda_1$ is normalized to be the identity matrix, $I_2$, the six structural parameters $\vartheta := (c_{11}, c_{12}, c_{21}, c_{22}, \lambda_{11,2}, \lambda_{22,2})'$, can be recovered uniquely from $\sigma_+$ by solving system (7).

This identification approach has been extended by Lanne and Lütkepohl (2008) to the case

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7In his Proposition 1, Rigobon (2003) provides a necessary and sufficient rank condition for the identification of bivariate systems. Using our notation, Rigobon’s (2003) Proposition 1 states that, if the parameters $c_{11}, c_{12}, c_{21}, c_{22}$ do not change across volatility regimes, they will be identified (up to normalization constraints) if and only if the two covariance matrices $\Sigma_{\varepsilon, 1}$ and $\Sigma_{\varepsilon, 2}$ are not proportional, i.e. $\Sigma_{\varepsilon, 2} \neq a\Sigma_{\varepsilon, 1}$, for some scalar $a$. This condition can be practically checked, after estimation, by verifying that $\text{det}(\Sigma_{\varepsilon, 2} - \frac{a}{a_{11,1}} \Sigma_{\varepsilon, 1}) \neq 0$, where $a := \frac{\sigma_{11,2}}{\sigma_{11,1}}$ is chosen to be the ratio of the variances of the first disturbance of the VAR in the two volatility regimes. We notice that Rigobon’s condition is satisfied when $CC' \neq CA_2C'$, or more simply, when $\lambda_{11,2} \neq \lambda_{22,2}$. In his Proposition 2, instead, Rigobon (2003) provides only a necessary order condition for identification. His setup, however, covers a class of dynamic models more general than the class of SVAR systems investigated in this paper.
of SVARs, where \( \dim(Z_t) = n > 2 \). Lanne and Lütkepohl (2008) exploit the algebraic result in Horn and Johnson (1985, Corollary 7.6.5), according to which the condition \( \Sigma_{\varepsilon,1} \neq \Sigma_{\varepsilon,2} \) guarantees the simultaneous factorization

\[
\Sigma_{\varepsilon,1} := PP' \quad \Sigma_{\varepsilon,2} := PVP'
\]  

(8)

where \( P \) is a \( n \times n \) non-singular matrix and \( V := \text{diag}(v_1, \ldots, v_n) \neq I_n \) is a diagonal matrix with \( v_i > 0, \; i = 1, \ldots, n \). Identification in this setup is achieved by setting \( C := C = P \) and \( \Lambda_2 := V \), where the choice \( C := P \) is unique except for sign changes if all \( v_i \)’s are distinct.8

2.3 Our contribution

The ‘purely statistical’ approach to the identification of SVARs put forth by Rigobon (2003) and Lanne and Lütkepohl (2008) has important implications for the transmission mechanisms of the shocks. In their framework, the structural break at time \( T_B \) affects only the VAR error covariance matrix, and the IRFs computed on the sub-samples \( Z_1, \ldots, Z_{T_B} \) and \( Z_{T_B+1}, \ldots, Z_T \) have the same time patterns.

In empirical macroeconomics, however, the situation in which the structural parameters do not vary across volatility regimes is the exception rather than the rule. Our paper contributes to this literature by relaxing the restrictive assumption that the changes in the volatility of the data have no impact on the transmission mechanisms of the shocks. In the next sections, we develop a new theoretical framework where both the variance of the structural shocks and the ‘simultaneity parameters’ contained in the matrix \( C \) may vary across volatility regimes and contribute to the identification of the SVAR. We show that the results in Rigobon (2003) and Lanne and Lütkepohl (2008) can be obtained as special cases of our approach.

When it is known that SVAR parameters, including the elements of \( C \), change at time \( T_B \), VAR practitioners typically deal with two distinct SVARs: one for the sub-sample \( Z_1, \ldots, Z_{T_B} \) and the other for the sub-sample \( Z_{T_B+1}, \ldots, Z_T \). Interestingly, we show that our approach leads to new identification schemes which can be fruitfully implemented without the need to recur to distinct SVARs.

3 Identification analysis

Consider the SVAR summarized in Eq.s (2)-(3) and assume that at time \( T_B, 1 < T_B < T \), the unconditional reduced form covariance matrix \( \Sigma_{\varepsilon} \) changes. We denote with \( \Sigma_{\varepsilon,1} \) and \( \Sigma_{\varepsilon,2} \) the covariance matrix before and after the break, respectively. Without any loss of generality, we focus on the case of a single break, i.e. two volatility regimes. Results are extended to the case

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8 Intuitively, the result follows from the observation that because of the two volatility regimes, Eq. (8) doubles the number of restrictions stemming from the symmetry of the covariance matrix, and these restrictions are enough to identify the elements of \( C \) and \( v_1, \ldots, v_n \).
of a finite number $s \geq 2$ of breaks in the Technical Supplement. In this section, we discuss the representation of the reference SVAR (Sub-section 3.1), and then introduce our main necessary and sufficient conditions for local identification in Proposition 1 (Sub-section 3.2). Finally, we present some new identification schemes, consistent with Proposition 1, by considering two examples taken from the empirical monetary policy literature (Subsection 3.3).

3.1 Representation

The reference reduced form VAR is given by:

$$Z_t = \Pi(t)W_t + \varepsilon_t, \quad \Sigma_\varepsilon(t) := E(\varepsilon_t\varepsilon'_t), \quad t = 1, ..., T \quad (9)$$

where

$$\Pi(t) := \Pi_1 \times 1 (t \leq T_B) + \Pi_2 \times 1 (t > T_B), \quad t = 1, ..., T \quad (10)$$

$$\Sigma_\varepsilon(t) := \Sigma_{\varepsilon,1} \times 1 (t \leq T_B) + \Sigma_{\varepsilon,2} \times 1 (t > T_B), \quad t = 1, ..., T, \quad (11)$$

$1(\cdot)$ is the indicator function, $\Pi_1 := (A_1, \Psi_1)$ and $\Pi_2 := (A_2, \Psi_2)$ are the $n \times f$ matrices containing the autoregressive coefficients before and after the break, respectively. As it stands, the specification in Eqs. (9)-(11) covers the case in which the structural break affects both the autoregressive coefficients and the error covariance matrix. The changes in the latter are of crucial importance in our approach. Thus we consider the following assumption.

**Assumption 1** [Change in the reduced form error covariance matrix] Given the VAR system in Eqs. (9)-(11), $T_B \geq f$ and $T - (T_B + 1) \geq f$, and it holds the condition

$$\sigma_{+1} := \text{vech}(\Sigma_{\varepsilon,1}) \neq \sigma_{+2} := \text{vech}(\Sigma_{\varepsilon,2}).$$

The main implication of Assumption 1 is that the two sub-samples $Z_1, ..., Z_{T_B}$ and $Z_{T_B+1}, ..., Z_T$ are characterized by two distinct regimes of volatility (and there are sufficient observations to estimate the VAR in each regime). In our framework, the autoregressive coefficients in Eq. (10) may change or may be equal at time $T_B$, i.e. both cases $\Pi_1 \neq \Pi_2$ and $\Pi_1 = \Pi_2$ are consistent with the identification analysis of the SVAR presented below. The break date, $T_B$, can be either known, a common assumption in macroeconomic analysis in which permanent (not recurring) events that lead to relevant institutional or behavioral changes are typically identified ex-post, or can be inferred from the data by applying any method suited to detect changes in the covariance matrix of the disturbances in multivariate regression models, see e.g. Qu and Perron (2007) and references therein.
Given Assumption 1, we consider the following counterpart of the structural specification in Eq. (3):

\[
\begin{align*}
\varepsilon_t &= C e_t & E(e_t e_t') &= \Lambda_1 & t \leq T_B \\
\varepsilon_t &= (C + Q) e_t & E(e_t e_t') &= \Lambda_2 & t > T_B
\end{align*}
\]

(12) (13)

where \( C \) and \( Q \) are two \( n \times n \) matrices of structural parameters, and \( \Lambda_1 \) and \( \Lambda_2 \) are two \( n \times n \) diagonal matrices with positive elements on the diagonal, which collect the variance of the structural shocks in the two volatility regimes. The non zero elements of the matrix \( Q \), instead, capture the changes in the structural parameters, if any, across the two regimes. In the following, it is convenient to normalize the variances of the structural shocks in the first regime to one, i.e. \( \Lambda_1 := I_n \). With this choice, the diagonal elements of the covariance matrix \( \Lambda := \Lambda_2 \) read as the variances of the structural shocks in the ‘post-break’ regime, relative to the ‘pre-break’ regime.

The relationships in Eq.s (12)-(13), including the normalization \( \Lambda_1 := I_n \), lead to the system of equations

\[
\begin{align*}
\Sigma_{\varepsilon,1} &= C C' \\
\Sigma_{\varepsilon,2} &= (C + Q) \Lambda (C + Q)'
\end{align*}
\]

that links the reduced to the structural form parameters of the SVAR. System (14)-(15) generalizes the specifications in Lanne and Lütkepohl (2008, 2010) (see the factorization in Eq. (8)), in several dimensions. In our setup, the difference in the reduced form VAR covariance matrices before and after the break, \( \Sigma_{\varepsilon,1} \neq \Sigma_{\varepsilon,2} \), is explained either by the change in the variances of the structural shocks before and after the break, i.e. \( \Lambda_1 := I_n \) and \( \Lambda_2 := \Lambda \neq I_n \), or by the different impact of the shocks across the volatility regimes if \( Q \neq 0_{n \times n} \), or, possibly, by a combination of these two factors. More formally, Lanne and Lütkepohl’s (2008) framework is obtained from system (14)-(15) when \( Q := 0_{n \times n} \). Indeed, for \( Q = 0_{n \times n} \), the system (14)-(15) is equivalent to that in Eq. (8), for \( C := P \) and \( \Lambda := V \).

Another interesting case nested in the system (14)-(15) obtains when \( \Lambda_2 := \Lambda := I_n \). In this situation, which is not consistent with the Lanne and Lütkepohl’s (2008) setup (see Bacchiocchi et al. (2013) for a detailed discussion), Assumption 1 implies that \( Q \neq 0_{n \times n} \), hence the change in the covariance matrix is solely explained by the changes in the structural parameters, i.e. one has \( \Sigma_{\varepsilon,1} = C C' \) in the first volatility regime, and \( \Sigma_{\varepsilon,2} = (C + Q)(C + Q)' \) in the second volatility regime.

The \( n(n + 1) \) symmetry restrictions provided by Eq.s (15)-(14) are not sufficient alone to identify the \( 2n^2 + n \) elements in the matrices \( C, Q \) and \( \Lambda \), hence it is necessary to add at least \( 2n^2 + n - n(n + 1) = n^2 \) restrictions on these matrices. Linear restrictions on the \( (2n^2 + n) \times 1 \)
vector $\vartheta := (\text{vec}(C)', \text{vec}(Q)',$ $w(\Lambda)')'$ can be imposed through a specification analogue to that in Eq. (4), i.e.

$$
\begin{pmatrix}
\text{vec}(C) \\
\text{vec}(Q) \\
w(\Lambda)
\end{pmatrix} :=
\begin{pmatrix}
S_C & S_I & 0_{n^2 \times a_\lambda} \\
0_{n^2 \times a_C} & S_Q & 0_{n^2 \times a_\lambda} \\
0_{n^2 \times a_C} & 0_{n^2 \times a_\lambda} & S_{\Lambda}
\end{pmatrix}
\begin{pmatrix}
\gamma \\
q \\
\lambda
\end{pmatrix} +
\begin{pmatrix}
s_C \\
s_Q \\
s_{\Lambda}
\end{pmatrix}
$$

where the $a \times 1$ vector $\psi := (\gamma', q', \lambda')'$ contains the $a = a_C + a_Q + a_\Lambda$ free elements of $C$, $Q$ and $\Lambda$, respectively. It is maintained that all components in the vector $\lambda$ are positive. The matrix $S$ and the vector $s$, of suitable dimensions, summarize the linear restrictions on $C$, $Q$ and $\Lambda$, and $S_I$ is a selection sub-matrix through which it is possible to imposes cross-restrictions on the elements of $C$ and $Q$.

The next sub-section shows that the relationships in Eq.s (14)-(15) and the restrictions in Eq. (16) can be used to identify the SVAR.

### 3.2 Main result

Consider the SVAR introduced in Sub-section 3.1, Assumption 1, and the set of restrictions in Eq.s (14)-(16). We denote with $\gamma_0$, $q_0$ and $\lambda_0$ the ‘true’ values of $\gamma$, $q$ and $\lambda$ respectively, and with $C_0$, $Q_0$ and $\Lambda_0$ the matrices obtained from Eq. (16) by replacing $\gamma$, $q$ and $\lambda$ with $\gamma_0$, $q_0$ and $\lambda_0$. Our main result is summarized in the next proposition.

**Proposition 1 [Identification of $C$ and $Q$ (and $\Lambda$)]** Assume that the data generating process belongs to the class of SVARs in Eq.s (9)-(11) and Eq.s (12)-(13), and that the matrices $C$, $\Lambda$ and $(C + Q)$ are non-singular, where $C$, $Q$ and $\Lambda$ are subject to the restrictions in Eq. (16). Under Assumption 1, the following statements hold:

(a) a necessary and sufficient rank condition for the SVAR to be locally identified is that the $n(n + 1) \times a$ matrix

$$
(I_2 \otimes D_n^+) \begin{pmatrix}
2(C \otimes I_n) & 0_{n^2 \times n^2} & 0_{n^2 \times n} \\
2((C + Q) \Lambda \otimes I_n) & 2 ((C + Q) \Lambda \otimes I_n) & ((C + Q) \otimes (C + Q)) U_n'
\end{pmatrix} S
$$

has full column rank $a := (a_C + a_Q + a_\Lambda)$ evaluated at $C := C_0$, $Q := Q_0$ and $\Lambda := \Lambda_0$; necessary order condition is

$$
a := (a_C + a_Q + a_\Lambda) \leq n (n + 1);
$$

(b) if $\Lambda_1 := \Lambda_2 := I_n$, necessary and sufficient condition for the SVAR to be identified is that
the \( n(n + 1) \times a \) matrix

\[
(I_2 \otimes D_n^+) \begin{pmatrix}
(C \otimes I_n) & 0_{n^2 \times n^2} \\
(C + Q) \otimes I_n & (C + Q) \otimes I_n
\end{pmatrix} \begin{pmatrix}
S_C & S_I \\
0_{n^2 \times a_C} & S_Q
\end{pmatrix}
\]

has full column rank \( a \) evaluated at \( C := C_0 \) and \( Q := Q_0 \); necessary order condition is

\[
a := (a_C + a_Q) \leq n(n + 1).
\]

**Proof.** See Appendix.

Point (a) of Proposition 1 deals with the most general case, i.e. the situation in which the identifying restrictions are placed simultaneously on \( C, Q \) (including possibly the cross-restrictions governed by the matrix \( S_I \)), and the covariance matrix \( \Lambda \).\(^9\) Point (b) of Proposition 1, instead, refers to the case in which the variances of the structural shocks \( e_t \) are kept constant (and equal to one) in both volatility regimes, and only the structural parameters change, so that the matrix of simultaneous relationships is \( C \) in the first volatility regime and \( (C + Q) \) in the second volatility regime. When in Eq. (18) \( a = n(n + 1) \), the SVAR is ‘exactly identified’, while is overidentified when \( a < n(n + 1) \).

If the specified matrices \( C, Q \) and \( \Lambda \) meet the requirements of Proposition 1, the log-likelihood of the SVAR can be maximized as described in the Technical Supplement. Moreover, if the SVAR is overidentified, the \( [n(n + 1) - a] \) overidentifying restrictions can be validated/rejected by computing a (quasi-) likelihood ratio (LR) test that compares the log-likelihood of the structural form and the log-likelihood of the reduced form. The (normalized) impulse responses implied by the identified SVAR are given by

\[
\Xi_{1,h} := [\psi_{1,lm,h}] := J'(\hat{A}_1)^h J\hat{C}, \ h = 0, 1, 2, \ldots \ \text{‘pre-change’ regime}
\]

\[
\Xi_{2,h} := [\psi_{2,lm,h}] := J'(\hat{A}_2)^h J(\hat{C} + \hat{Q}), \ h = 0, 1, 2, \ldots \ \text{‘post-change’ regime}
\]

where \( \hat{A}_1 \) and \( \hat{A}_2 \) are the companion matrices in the two volatility regimes, as described in Eq. (6), and \( \hat{C} \) and \( \hat{Q} \) denote counterparts of \( C \) and \( Q \) such that the rank conditions of Proposition 1 are fulfilled. Note that irrespective of whether \( \hat{A}_1 = \hat{A}_2 \) or \( \hat{A}_1 \neq \hat{A}_2 \), the two sets of population impulse responses in Eq. (21) and Eq. (22) differ across volatility regimes when \( Q \neq 0_{n \times n} \). The coefficient \( \psi_{i,lm,h} \) captures the response of variable \( l \) to a one-time impulse in variable \( m \), \( h \) periods before, in the volatility regime \( i \).

\(^9\)The rank conditions in Proposition 1 can be checked numerically ex-post, at the maximum likelihood estimates, or prior to estimation by using algorithms no more complicated than the algorithms originally proposed by Giannini (1992) for SVARs and successively suggested by Iskrev (2010) for dynamic stochastic general equilibrium models.
3.3 Identification schemes: some examples

Proposition 1 opens up new identification schemes for SVARs that we discuss with some examples taken from the monetary policy literature. The models we discuss below would not be identified through the ‘standard’ identification approach summarized in Sub-section 2.1, or by considering two distinct SVARs on the sub-samples $Z_1, \ldots, Z_{T_B}$ and $Z_{T_B+1}, \ldots, Z_T$.

**Example 1 [‘DSGE-consistent SVAR’]** Consider the three-variable monetary policy SVAR in which $Z_t := (\tilde{y}_t, \pi_t, R_t)'$ ($n:=3$), where $\tilde{y}_t$ is a measure of the output gap, $\pi_t$ the inflation rate and $R_t$ a nominal policy interest rate. Imagine that a structural break changes the error covariance matrix at time $T_B$, and that the structural specification in Eqs (12)-(13) specializes to

$$
\begin{pmatrix}
  \varepsilon_{1,t}^{\prime} \\
  \varepsilon_{2,t}^{\prime} \\
  \varepsilon_{3,t}^{\prime} \\
  \varepsilon_t
\end{pmatrix} =
\begin{pmatrix}
  c_{11} & c_{12} & c_{13} \\
  c_{21} & c_{22} & c_{23} \\
  c_{31} & c_{32} & c_{33}
\end{pmatrix}
+ \begin{pmatrix}
  q_{11} & 0 & 0 \\
  0 & q_{22} & 0 \\
  0 & 0 & q_{33}
\end{pmatrix} \times 1(t>T_B)
\begin{pmatrix}
  e_{1,t} \\
  e_{2,t} \\
  e_{3,t}
\end{pmatrix}
$$

(23)

where the covariance matrices of the structural shocks are the same across volatility regimes, i.e $\Lambda_1 = \Lambda_2 := I_3$. It is here ‘natural’ to interpret $e_{3,t}$ as the ‘monetary policy shock’, $e_{1,t}$ as the ‘output shock’ and $e_{2,t}$ as the ‘inflation shock’. Apparently, the SVAR in Eq. (23) is ‘close’ to the one based on the factorization in Eq. (8). However, despite the $C$ matrix is ‘full’ in both cases, in Eq. (23) the instantaneous impact of the shock $e_{j,t}$ on the variable $Z_{j,t}$, $j=1,2,3$ varies from $c_{jj}$ in the first volatility regime to $c_{jj} + q_{jj}$ in the second volatility regime. The specification in Eq. (23) is interesting because it can be to some extent related to the debate about the consistency between SVAR analysis and dynamic stochastic general equilibrium (DSGE) modeling, see Bacchiocchi et al. (2013). As is known, small-scale new-Keynesian DSGE models of the type discussed, among many others, in e.g. Lubik and Schorfheide (2004) and Carlstrom et al. (2009), typically admit an immediate reaction of output and inflation to monetary policy impulses, while ‘conventional’ triangular (Cholesky-based) SVARs feature a lag in such reactions. The existing empirical evidence seems to suggest that monetary policy shocks exert a non-zero instantaneous impact on macroeconomic variables like prices and output. For instance, by employing their ‘DSGE-VAR’ approach, Del Negro et al. (2007) find Cholesky-based SVARs to be implausible due to the very likely immediate reaction of output to a policy shock. Likewise, Faust et al. (2004) show that the zero response of prices to a monetary policy shock imposed by Cholesky-based SVARs is not supported by the data when disturbances are inferred using high frequency futures data in a two-step procedure. Thus, under the null of a valid DSGE model, triangular SVARs offer a misspecified representation of monetary policy shocks and their propagation, and can produce price puzzles...
and muted responses of inflation and the output gap to monetary shocks, see Castelnuovo and Surico (2010), Castelnuovo (2012b) and Bacchiocchi et al. (2013). In other words, given $\varepsilon_t := Ce_t$, the $C$ matrix must be ‘full’ with highly restricted non-zero coefficients for the SVAR to be consistent with the predictions of a DSGE model. Eq. (23) suggests that an identified SVAR featuring a ‘full’ matrix $C$ can be obtained on condition that the response on impact of $Z_{j,t}$ to $e_{j,t}$ varies across volatility regimes. It is worth remarking, however, that in our framework the SVAR does not embody the whole set of restrictions on the VAR lag structure implied by DSGE models under rational expectations, and that the matrix $C$ does not feature the highly nonlinear cross-equation restrictions implied by these models. With a slight abuse of language, we denote the system defined by Eq. (23) as ‘DSGE-consistent SVAR’, where by this term we mean a SVAR which, aside from the cross-equation restrictions, features a ‘full’ matrix $C$, so that all shocks hitting the modeled economy are allowed to affect all variables contemporaneously, as typically predicted by DSGE models.

**Example 2 [SVAR with changing policy reaction function]** Consider the same three-variable monetary policy SVAR of the previous example, and the structural specification

$$
\left( \begin{array}{c}
\varepsilon_{1,t} \\
\varepsilon_{2,t} \\
\varepsilon_{3,t} \\
\varepsilon_t
\end{array} \right) := \left( \begin{array}{ccc}
c_{11} & 0 & c_{13} \\
0 & c_{22} & c_{23} \\
c_{31} & c_{32} & c_{33}
\end{array} \right) + \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
q_{31} & q_{32} & q_{33}
\end{array} \right) C + \left( \begin{array}{c}
e_{1,t} \\
e_{2,t} \\
e_{3,t} \\
e_t
\end{array} \right) \mathbf{1} (t > T_B)
$$

(24)

where the covariance matrix of the structural shocks is now given by

$$
E(e_te_t') := \left( \begin{array}{c}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) + \left( \begin{array}{ccc}
\lambda_{11.2} - 1 & 0 & 0 \\
0 & \lambda_{22.2} - 1 & 0 \\
0 & 0 & 0
\end{array} \right) \mathbf{1} (t > T_B),
$$

(25)

hence it is equal to the identity matrix ($\Lambda_1 := I_3$) in the first volatility regime, and to

$$
\Lambda_2 := \lambda := \left( \begin{array}{ccc}
\lambda_{11.2} & 0 & 0 \\
0 & \lambda_{22.2} & 0 \\
0 & 0 & 1
\end{array} \right)
$$

in the second volatility regime. This model describes an heteroskedastic SVAR in which the change in the variance of the disturbances at time $T_B$ is explained by (i) a break in the variances of the output gap ($e_{1,t}$) and the inflation ($e_{2,t}$) shocks, which change from 1 to $\lambda_{11.2}$ and $\lambda_{22.2}$, respectively, and (ii) a change in the parameters governing the response
of the policy reaction function to the structural shocks, see the last row of the matrix $Q$. In particular, the response on impact of the nominal short term interest rate $R_t$ to $e_{1,t}$, $e_{2,t}$ and $e_{3,t}$, is postulated to change from the levels $c_{3j}$ in the ‘pre-break’ regime, to the levels $c_{3j} + q_{3j}$ in the ‘post-break’ regime, $j = 1, 2, 3$. Moreover, this system shares with the ‘DSGE-consistent SVAR’ presented in the Example 1 the fact that it is non-triangular (non-recursive), because the monetary policy shock may have an instantaneous impact on the macroeconomic variables $\tilde{y}_t$ and $\pi_t$ in both volatility regimes, as implied by the non-zero specification of the last column of the matrix $C$. As it stands, however, the specification in Eq.s (24)-(25) meets the necessary order condition of Proposition 1 (indeed $a_C := \text{dim}(\gamma) = 7$, $a_Q := \text{dim}(q) = 3$, $a_{\Lambda} := \text{dim}(\lambda) = 2$, so that there are $a = 12$ structural parameters and $n(n + 1) = 12$ estimable moments in the covariance matrices $\Sigma_{\varepsilon,1}$ and $\Sigma_{\varepsilon,2}$), but does not satisfy the necessary and sufficient rank condition in Eq. (17). (Over)identification is achieved by imposing e.g. the (testable) restriction $q_{33} = 0$ in the $Q$ matrix. This restriction maintains that the response of the short term interest rate to monetary policy shocks is invariant across the two volatility regimes. The specification in Eq.s (24)-25) will be estimated and tested in the next section, using U.S. quarterly data.

4 Empirical illustration

In this section, we apply the identification rules derived in Section 3 to estimate a small monetary policy SVAR using U.S. quarterly data. As in Lanne and Lütkepohl (2008), we identify the shocks by exploiting the change in volatility that occurred across several macroeconomic time series in the transition from the ‘Great Inflation’ to the ‘Great Moderation’ regimes, documented, among many others, in McConnell and Perez-Quiros (2002) and Boivin and Giannoni (2006). Our SVAR is based on the vector $Z_t := (\tilde{y}_t, \pi_t, R_t)'$ $(n := 3)$, where $\tilde{y}_t$ is a measure of the output gap, $\pi_t$ the inflation rate and $R_t$ a nominal policy interest rate. We deal with quarterly data, sample 1954.q3-2008.q3 (including initial values). The measure of real activity, $\tilde{y}_t$, is the Congressional Budget Office (CBO) output gap, constructed as percentage log-deviations of real GDP with respect to CBO potential output. The measure of inflation, $\pi_t$, is the annualized quarter-on-quarter GDP deflator inflation rate, while the policy instrument, $R_t$, is the Federal funds rate (average of monthly observations). The data were collected from the website of the Federal Reserve Bank of St. Louis.

We discuss the identification and estimation of the SVAR for $Z_t := (\tilde{y}_t, \pi_t, R_t)'$ in Sub-section 4.1, and summarize some robustness checks in Sub-section 4.2.
4.1 A small monetary policy SVAR

In line with the empirical literature on the ‘Great Moderation’, we divide the postwar period 1954.q3-2008.q3 into two sub-samples: the ‘pre-Volcker’ period, 1954.q3-1979.q2, and the ‘Great-Moderation’ period, 1979.q3-2008.q3. This choice is consistent with Boivin and Giannoni (2006).10 In our notation, \( T_B := 1979.q2 \), and hereafter this date will be treated as known. Our statistical tests, presented below, confirm that the two sub-periods 1954.q3-1979.q2 and 1979.q3-2008.q3 can be regarded as two periods characterized by different volatilities. The modeled reduced form VAR is a system with six lags \( (k := 6) \) and a constant. The VAR lag order is obtained by combining LR-type reduction tests with standard information criteria. Table 1 reports the estimated covariance matrices of the VAR and some (multivariate) residual diagnostic tests relative to the entire period and the two sub-periods, respectively. The Technical Supplement motivates our choice of treating the VAR for \( Z_t := (\tilde{y}_t, \pi_t, R_t)' \) as a stationary system, and checks the robustness of our specification to a different choice of the VAR lag order.

We test for the occurrence of a break at time \( T_B := 1979.q2 \) in the reduced form coefficients \( \theta := (\pi', \sigma', \sigma_+)' \) of the VAR, in particular in the error covariance matrix \( \sigma_+ \). We first apply a standard Chow-type (quasi-)LR test for the (joint) null \( H_0^1: \theta_1 = \theta_2 = \theta \) against the alternative \( H_1^1: \theta_1 \neq \theta_2 \), where \( p := \text{dim} (\theta) = 63 \). The results (Table 1) suggest that \( H_0^1 \) is strongly rejected because the (quasi-)LR test is equal to \( LR := -2[719.98-(322.86+488.96)] = 183.68 \) and has a p-value of 0.000 (taken from the \( \chi^2(63) \) distribution). We then test the null \( H_0^{(1)}: \sigma_{+1} = \sigma_{+2} = \sigma_+ \) vs \( H_1^{(1)}: \sigma_{+1} \neq \sigma_{+2} \) while maintaining the assumption that \( \pi_1 = \pi_2 = \pi \). The results of the computed (quasi-)LR test also lead us to strongly reject the null \( H_0^{(1)} \). We find formal support to the hypothesis that our ‘pre-Volcker’ and ‘Great moderation’ samples are characterized by two distinct volatility regimes.

We next move to the identification of the structural shocks, considering the schemes discussed in the two examples of Sub-section 3.3. We denote with \( \mathcal{M}_1 \) the exactly identified ‘DSGE-consistent SVAR’ in the Example 1, see Eq. (23), and with \( \mathcal{M}_2 \) the ‘SVAR with changing policy reaction function’ in the Example 2, see Eq.s (24)-(25), including the overidentifying restriction \( q_{33} = 0 \) in the \( Q \) matrix. Given the vector of structural shocks \( e_t := (e_t^{\tilde{y}}, e_t^\pi, e_t^R)' \), we call \( e_t^R \) the ‘monetary policy shock’, \( e_t^{\tilde{y}} \) the ‘output shock’ and \( e_t^\pi \) the ‘inflation shock’.

The (quasi-)ML estimates of the structural parameters of \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are summarized in Table 2, which also reports the log-likelihood associated with each model and the (quasi-)LR test for the overidentifying restriction for \( \mathcal{M}_2 \). Many studies based on SVARs typically find

\[ 10 \text{We are aware that many other choices for } T_B \text{ are equally possible: an alternative would be to start the second period in 1984.q1 as in e.g. McConnell and Perez-Quiros (2002). It is worth observing that the period 1979.q3-2008.q3 includes the three-year window, 1979-1982, known as the ‘Volcker experiment’, during which the Federal Reserve implemented monetary policy actions by dealing with non-borrowed reserves, more than with the federal fund rate. As a result, the IRFs we compute on the ‘post-Volcker’ period (see Figure 1 and Figure 2 below) might be affected by the course of this ‘non-standard’ policy. Unfortunately, the sub-period 1979.q3-1984.q1 is not long enough to allow for the consideration of two break dates and three potential volatility regimes in the empirical analysis.} \]
that U.S. monetary policy shocks have had a much smaller impact on output gap and inflation since the beginning of the 1980s. Overall, the results in Table 2 seem to confirm such evidence. In addition, we detect significant changes to the structural parameters in the move from the ‘pre-Volcker’ to the ‘post-Volcker’ period, because the specified elements in the $Q$ matrices are found to be highly significant in both estimated SVARs. However, we also notice that as concerns the model $M_1$, the sign of the estimated parameters relative to the ‘post-Volcker’ period (i.e. the elements of the matrices $\hat{C} + \hat{Q}$, last column of Table 2) are not consistent with what predicted by a small monetary policy DSGE model. This result admits at least two explanations. First, the estimated SVAR $M_1$ does not feature the cross-equation restrictions implied by a monetary DSGE model (see the discussion in the Example 1). Second, the system might omit, as it stands, important transmission mechanisms, see Bacchiocchi et al. (2013).

The results stemming from the SVAR $M_2$ are more interesting. Table 2 shows that model $M_2$ is strongly supported by the data by the (quasi-)LR test for the overidentifying restriction, which has a p-value equal to 0.26. We recall that the SVAR $M_2$ maintains that both the change in the variances of the structural shocks and the change in the response of monetary policy to the macroeconomic shocks $e^\hat{y}_t$ and $e^\pi_t$ contribute to explain the change in the volatility of the data. Figure 1 displays, for both volatility regimes, the IRFs implied by $M_2$ relative to the monetary policy shock, $e^R_t$, with associated 95% (asymptotic) confidence interval, over a horizon of 20 periods. To improve the comparability of the IRFs, we have normalized the quarterly response on impact of the Federal funds rate $R_t$ to a monetary policy shock $e^R_t$ at the value 0.25 in both volatility regimes. The pattern of the two sets of impulse responses reveals the change in the monetary policy conduct. The key result from the comparison of the ‘pre-Volcker’ period (left column) and the ‘post-Volcker’ period (right column) in Figure 1 is that the effect of a monetary policy shock was stronger before the 1980s.11 Figure 2 displays instead the response of the Federal funds rate to the shocks $e^\hat{y}_t$ and $e^\pi_t$, respectively, for both samples. While the sensitivity of the short term nominal interest rate to the two shocks seems to be weak prior to the 1980s, the Fed’s responsiveness to these two shocks is clear cut in the ‘post-Volcker’ period. According to a large (but much debated) strand of the literature, this evidence reflects the switch to a more aggressive (‘active’) policy intended to rule out the possibility of sunspot fluctuations induced by self-fulfilling expectations, see e.g. Clarida et al. (2000). The interesting fact is that the change in the monetary policy conduct is identified in our framework in spite of the sharp reduction in the variances of the structural shocks $e^\hat{y}_t$ and $e^\pi_t$, which are equal to 1 in the ‘pre-Volcker’ regime, and to 0.334 and 0.360, respectively, in the ‘Great Moderation’ sample. Hence, our identification approach and estimation results seem to

11A further remarkable fact that emerges from the impulse responses in Figure 1 is the absence of the ‘price puzzle’ in the ‘post-Volcker’ period. This evidence, which is also documented in e.g. Barth and Ramey (2001), Hanson (2004), Boivin and Giannoni (2006) and Castelnuevo and Surico (2010), supports the view that the ‘price puzzle’ phenomenon is far more evident in situations in which the central bank responds weakly to inflationary dynamics.
suggest that both the ‘good luck’ hypothesis (the reduction of the volatility of the shocks) and the ‘good policy’ hypothesis (a more aggressive response of monetary policy to macroeconomic conditions, in particular to inflation) have contributed to the ‘Great Moderation’.

4.2 Robustness of the results

We conduct some experiments to check the robustness of our results to the specification of a different VAR lag order, the use of output growth in place of the output gap, and the inclusion of monetary balances. The details of these analyses are reported in the Technical Supplement, and unequivocally indicate that the core results discussed in the previous sub-section are indeed robust. We briefly summarize the main findings.

**Different lag order.** We consider a different dynamic specification based on a VAR with four lags, as suggested by many empirical contributions in the literature, see, among many others, Christiano _et al._ (2005). We strongly reject the null hypothesis of constant covariance matrices before and after time \( T_B := 1979.q2 \), and the set of IRFs implied by the SVAR \( \mathcal{M}_2 \) do not differ qualitatively from the IRFs reported in Figure 1 and Figure 2.

**Output growth.** The output gap is largely used in the empirical literature and should be preferred, in our context, to other measures of economic activity for reasons discussed in e.g. Giordani (2004). However, it might reasonably be affected by measurement errors. The natural alternative is to replace the output gap with real output growth, \( \Delta y_t \). The estimation of the SVAR \( \mathcal{M}_2 \) based on \( Z_t^* := (\Delta y_t, \pi_t, R_t)' \) shows that there are still two distinct regimes of volatility before and after \( T_B := 1979.q2 \), and that the implied IRFs do not differ qualitatively from the IRFs reported in Figure 1 and Figure 2.

**Omitted variables: the role of money.** A variety of recent empirical studies suggest that omitting money balances in the analysis of the monetary transmission mechanisms can produce severely distorted inference, see, *inter alia*, Canova and Menz (2009), Favara and Giordani (2009) and Castelnuovo (2012). A thorough investigation of the role of monetary aggregates in the dynamics of U.S. business cycle and the actual transmission mechanisms at work goes well beyond the scopes of our paper. We limit our attention to check the robustness of our results to considering the identification and estimation of a SVAR based on \( Z_t^{**} := (\tilde{y}_t, \pi_t, R_t, \Delta mr_t)' \), where \( \Delta mr_t \) is the growth rate of real money balances. The variable \( mr_t := m_t - p_t \) is constructed by taking the log of the M2 money stock divided by the GDP deflator (source: website of the Federal Reserve Bank of St. Louis). Also in this case, we strongly reject the null hypothesis of constant covariance matrices for the VAR for \( Z_t^{**} := (\tilde{y}_t, \pi_t, R_t, \Delta mr_t)' \) before and after time \( T_B := 1979.q2 \). To identify the SVAR, we exploit part of the (non-triangular) identifying restrictions already used for the SVAR \( \mathcal{M}_2 \). In addition, we postulate that the response on impact of real money growth to the output gap shock is non-zero on the ‘pre-Volcker’ period and is zero on the ‘Great Moderation’ period, as a consequence of the increasing attention of firms towards financial
markets. Overall, our empirical results show that even controlling for real money growth, the IRFs obtained with our baseline three-equation SVAR $M_2$ in Figure 1 and Figure 2 remain qualitatively unchanged.

5 Conclusions

A recent strand of the literature makes use of the heteroskedasticity found in the data to identify SVARs. The assumption maintained in this literature is that the structural parameters remain constant when the VAR covariance matrix changes, consistently with the idea that structural changes offer identifying power only if some parameters do not change.

In this paper, we have relaxed the assumption that all structural parameters are invariant to volatility regimes. We have derived novel necessary and sufficient rank conditions which nest the result of other authors about the identification of SVARs via changes in volatility. We have illustrated the usefulness of our approach by focusing on a small-scale monetary policy SVAR model estimated using U.S. quarterly data. Overall, our results support the view that monetary policy has become more effective at stabilizing the economy since the 1980s.

Appendix: Proof of Proposition 1

(a) We write the mapping between the reduced- and structural-form parameters in Eqs (14)-(15) in the form

$$\sigma_+^* = g(\psi)$$

where $\sigma_+ := (\sigma_+^1, \sigma_+^2)'$ and $g(\cdot)$ is a nonlinear differentiable vector function. Given the constraints in Eq. (16) and following Rothenberg (1971), necessary and sufficient condition for local identification is that the $n(n+1) \times a$ Jacobian matrix

$$\frac{\partial \sigma_+^*}{\partial \psi} = \frac{\partial \sigma_+^*}{\partial \vartheta} \times \frac{\partial \vartheta}{\partial \psi} = \frac{\partial \sigma_+^*}{\partial \vartheta} \times S$$

has full column rank $a := a_C + a_Q + a_\Lambda$, evaluated a $\psi_0$. The necessary order condition $a \leq n(n+1)$ is therefore obvious.

To compute the $n(n+1) \times (2n^2 + n)$ matrix $\frac{\partial \sigma_+^*}{\partial \vartheta}$, it is convenient to apply the first differential and standard matrix algebra rules to the system in Eqs. (14)-(15), obtaining

$$2N_n (C \otimes I_n) vec dC = 0_{n^2 \times 1}$$

$$2N_n ((C + Q) \Lambda \otimes I_n) vec dC + 2N_n ((C + Q) \Lambda \otimes I_n) vec dQ + ((C + Q) \otimes (C + Q)) vec d\Lambda = 0_{n^2 \times 1}.$$
Using the results in Magnus (1988, pages 109-110), the last addend of system (28) can be written as

\[
[(C + Q) \otimes (C + Q)] \text{vec} \, d\Lambda = [(C + Q) \otimes (C + Q)] \, N_n U_n^\prime w \, (d\Lambda)
\]

where we have used the property \( U_n = U_n N_n \) and the symmetry of \( N_n \). Hence, the expression simplifies to

\[
N_n \left[ 2 \left( (C + Q) \Lambda \otimes I_n \right) \text{vec} \, dC + 2 \left( (C + Q) \Lambda \otimes I_n \right) \text{vec} \, dQ + \left( (C + Q) \otimes (C + Q) \right) U_n^\prime w \, (d\Lambda) \right] = 0_{n^2 \times 1}.
\]

However, \( N_n = D_n D_n^+ \), implying that among the set of \( n^2 \) equations of the system (27)-(28), only \( n(n+1)/2 \) are linearly independent, and in particular, those given by

\[
2D_n^+ (C \otimes I_n) \text{vec} \, dC = 0_{1 \times n^2(1+n)}
\]

\[
2D_n^+ ((C + Q) \Lambda \otimes I_n) \text{vec} \, dC + 2D_n^+ ((C + Q) \Lambda \otimes I_n) \text{vec} \, dQ + \left( (C + Q) \otimes (C + Q) \right) U_n^\prime w \, (d\Lambda) = 0_{1 \times n^2(1+n)}.
\]

Therefore the derivative \( \frac{\partial \sigma^*}{\partial \varphi} \) can be written as

\[
\frac{\partial \sigma^*}{\partial \varphi} = \begin{pmatrix} D_n^+ & 0_{1 \times n^2(1+n)} \\ 0_{1 \times n^2(1+n)} & D_n^+ \end{pmatrix} \times \begin{pmatrix} 0_{n^2 \times 1} \\ 2((C + Q) \Lambda \otimes I_n) \\ 2((C + Q) \Lambda \otimes I_n) \end{pmatrix}
\]

and coming back to Eq. (26), the result is obtained.

(b) When \( \Lambda:=I_n \), we repeat the same derivations as above, with the difference that only the parameters in \( C \) and \( Q \) are involved. In this case, the matrix \( \frac{\partial \sigma^*}{\partial \varphi} \) reduced to the first two block columns of Eq. (29) and the result is obtained. \( \blacksquare \)
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TABLE 1
Estimated covariance matrices and diagnostic tests from the VAR with six lags. Break date $T_B := 1979.q2$

<table>
<thead>
<tr>
<th>Period</th>
<th>Start-End</th>
<th>$T$</th>
<th>$\dim(\theta)$</th>
<th>$\sigma_{\varepsilon}$</th>
<th>Log-Likelihood</th>
<th>$J_B N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall period</td>
<td>1954.q3-2008.q3</td>
<td>211</td>
<td>63</td>
<td>$\begin{pmatrix} 0.762 &amp; -0.012 &amp; 0.037 \ 0.273 &amp; 0.007 &amp; 0.191 \end{pmatrix}$</td>
<td>719.98</td>
<td>186.27</td>
</tr>
<tr>
<td>‘Great Moderation’</td>
<td>1954.q3-1979.q2</td>
<td>94</td>
<td>63</td>
<td>$\begin{pmatrix} 0.989 &amp; -0.021 &amp; 0.036 \ 0.314 &amp; 0.007 &amp; 0.152 \end{pmatrix}$</td>
<td>322.86</td>
<td>14.525</td>
</tr>
<tr>
<td>‘post-Volcker’</td>
<td>1979.q3-2008.q3</td>
<td>117</td>
<td>63</td>
<td>$\begin{pmatrix} 0.562 &amp; 0.010 &amp; 0.045 \ 0.204 &amp; 0.007 &amp; 0.194 \end{pmatrix}$</td>
<td>488.96</td>
<td>50.91</td>
</tr>
</tbody>
</table>

NOTES: $LM_{AR5}$ is the Lagrange Multiplier vector test for the absence of residuals autocorrelation against the alternative of autocorrelated VAR disturbances up to lag 5; $J_B N$ is the Jarque-Bera multivariate test for Gaussian disturbances. Number in brackets are p-values.
\textbf{TABLE 2.} \newline Estimated SVARs with break date $T_B:=1979.43$ on U.S. quarterly data $Z_t:=(\hat{y}_t, \pi_t, R_t)'$, $e_t:=(e_t^y, e_t^\pi, e_t^R)'$

<table>
<thead>
<tr>
<th>Model:</th>
<th>$C(t):=C + Q \times 1 \ (t &gt; T_B)$, $t = 1, \ldots, T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{M}_1$</td>
<td>$(\hat{C} + \hat{Q}), t &gt; T_B$</td>
</tr>
<tr>
<td>$\hat{C}$</td>
<td>\begin{pmatrix} 0.876 &amp; -0.045 &amp; -0.105 \ (0.065) &amp; (0.055) &amp; (0.054) \end{pmatrix}</td>
</tr>
<tr>
<td>$\hat{Q}$</td>
<td>\begin{pmatrix} -0.374 \ (0.073) \end{pmatrix}</td>
</tr>
<tr>
<td>$(\hat{C} + \hat{Q})$, $t &gt; T_B$</td>
<td>\begin{pmatrix} 0.502 &amp; 0.045 &amp; 0.105 \ -0.020 &amp; 0.147 &amp; 0.115 \ 0.049 &amp; -0.072 &amp; 0.155 \end{pmatrix}</td>
</tr>
</tbody>
</table>

Log-Likelihood = 811.81 exact identification

$\mathcal{M}_2$ | \begin{pmatrix} 0.803 & -0.058 \\ (0.064) & (0.102) \end{pmatrix} |
| $\hat{C}$ | \begin{pmatrix} 0.263 & -0.100 \\ (0.031) & (0.051) \end{pmatrix} |
| $\hat{Q}$ | \begin{pmatrix} 0.106 & 0.114 \\ (0.048) & (0.068) \end{pmatrix} |
| $(\hat{C} + \hat{Q})$, $t > T_B$ | \begin{pmatrix} 0.803 & -0.058 \\ 0.263 & -0.100 \\ 0.148 & 0.181 & 0.112 \end{pmatrix} |

$\hat{\Lambda} = \begin{pmatrix} 0.334 \\ (0.067) \\ 0.360 \\ (0.117) \end{pmatrix}$ \newline \begin{pmatrix} 1 \end{pmatrix}

Log-Likelihood = 811.36 LR test = 0.90 \begin{pmatrix} 0.26 \end{pmatrix}

\textbf{NOTES:} Standard errors in parenthesis, p-values in squared brackets. The columns of the matrix $(C+Q)$ have been normalized such that the elements on the main diagonal are positive. Empty entries correspond to zeros. The reduced form associated with the estimated SVAR is a VAR with six lags.
FIGURE 1. Impulse responses of the variables in $Z_t$ to a same-size monetary policy shock $e_t^R$ with 95% (analytic) confidence bands, based on the SVAR model $M_2$ estimated in Table 2.

FIGURE 2. Impulse responses of $R_t$ to same-size output and inflation shocks $e_t^\tilde{y}$ and $e_t^\pi$ with 95% (analytic) confidence bands based on the SVAR model $M_2$ estimated in Table 2.