Sets, Proofs and Functions

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Takeaways

- Sets and set operations.
- De Morgan's laws.
- Logical quantifiers and operators.
- Implication.
- Methods of proof, deduction, induction, contradiction.
- Functions.

Introduction

- A set is a collection of elements. For example

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\},$$

which is the set of integers. We denote an element of $\mathbb{Z}$ by z and write $z \in \mathbb{Z}$. Eg. $2 \in \mathbb{Z}$.

- The set in the above example is *countably infinite*.
- If all the elements of some set $\mathbb{N}$ belong to $\mathbb{Z}$, then $\mathbb{N}$ is called a subset of $\mathbb{Z}$. Denoted by

$$\mathbb{N} \subset \mathbb{Z} \Leftrightarrow (z \in \mathbb{N} \Rightarrow z \in \mathbb{Z})$$

- For example

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\},$$

which is the set of natural numbers.

Set Operations I

Let U denote the universe or universal set and let the following sets be in U. We define the following properties

- *Union*: $A \cup B = \{x \in U; x \in A \text{ or } x \in B\}$
- *Intersection*: $A \cap B = \{x \in U; x \in A \text{ and } x \in B\}$
- *Complement*: $A^c = \{x \in U; x \notin A\}$. Hence, $A \cap A^c = \emptyset$
- *Difference*: $A \setminus B = \{x \in U; x \in A \text{ and } x \notin B\}$.

Example

Show in a Venn diagram that $A \setminus B = A \cap B^c$.

Set Laws

Theorem

Let A, B and C be three subsets of U. Then the following properties hold:

- *Commutative* law: $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
- *Associative* law: $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$ and $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$.
- *Distributive* law: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ and $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Class Exercise

Show these properties by means of a Venn diagram.

De Morgan's Laws

Theorem

1. $(A \cup B)^c = A^c \cap B^c$.
2. $(A \cap B)^c = A^c \cup B^c$.

Class Exercise

Show De Morgan's laws in a Venn diagram and determine how would they would (notationally) generalize to a family of sets $A = \{A_i; i \in I\}, I \subseteq \mathbb{N}, (\mathbb{N} = \mathbb{Z}^+)$ in U?

Logical Quantifiers

If each $x \in X$ has some property P we can define the set P_T consisting of the elements in X for which property P holds, $P(x)$. This is the set

$$P_T = \{x \in X; P(x)\}.$$

This leads to the following quantifiers

Logical Quantifiers

- $\forall x \in X; P(x)$, $P(x)$ is true for all $x \in X$.
- $\exists x \in X; P(x)$ there exists at least one $x \in X$ such that $P(x)$ is true.

Logical Operators

In addition we have the following logical operators

Logical Operators

- $(\neg P)_T = \{x \in X; \neg P(x) \text{ is true}\}$.
- $(P \wedge Q)_T = \{x \in X; P(x) \text{ and } Q(x)\}$.
- $(P \vee Q)_T = \{x \in X; P(x) \text{ or } Q(x)\}$.

Class Exercise

What is the relationship between the operators $\neg$, $\wedge$ and $\vee$ and the set operators above?

Implication

If property P implies property Q we write $P \Rightarrow Q$. This means that

- P is a sufficient condition for Q
- Q is a necessary condition for P.

The above notation is equivalent to

$$\forall x \in P_T, Q(x) \quad \text{or} \quad P_T \subseteq Q_T$$

If $P \Rightarrow Q$ and $Q \Rightarrow P$, then P and Q are equivalent and we write $P \Leftrightarrow Q$.

Example

For example, as we will see in the Calculus section

$$\text{differentiability} \Rightarrow \text{continuity}.$$

Methods of Proof: Deduction

We prove the statement $P \Rightarrow Q$ or $P \Leftrightarrow Q$ by assuming that P holds and then verifying if Q is true.

Example

Prove De Morgan's first law: $(A \cup B)^c = A^c \cap B^c$

Class Exercise

Prove that $A \setminus B = A \cap B^c$.

Methods of Proof: Induction I

Axiom

Let P be a property that natural numbers, $\mathbb{N}$ may or may not have. If

1. $\exists n_0 \in \mathbb{N} : P(n_0)$ holds and
2. $\forall n \in \mathbb{N}, \quad P(n) \Rightarrow P(n+1)$,

then $P(n)$ holds for all $n \in \mathbb{N}, n \geq n_0$.

Methods of Proof: Induction II

Example

Prove by induction $\sum_{m=1}^{k} m = \frac{k(k+1)}{2}$.

Class Exercise

Prove by induction $\sum_{m=1}^{k} m^3 = \left(\sum_{m=1}^{k} m \right)^2$. **Hint**: use the result from the example.

Methods of Proof: Contradiction

If we show that $P \not\Rightarrow Q$ does not hold (contradiction), then it must be that $P \Rightarrow Q$.

Class Exercise

Prove by contradiction: *There is no greatest even integer*.

Methods of Proof in Economics Textbooks

- It is immediate that...
- It is trivial to see that...
- Obviously...

Functions I

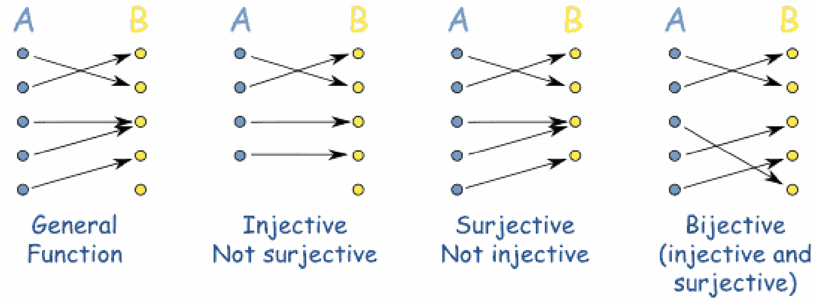
Definition

A *function* from the set X to Y written $f : X \to Y$ or $y = f(x)$ is a rule that assigns to each element of X a unique element of Y.

- The domain of a function is the set X and its range is the set $f(X)$
- A function is *injective* or *one-to-one* if it always assigns different images for every $x \in X$, i.e. $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$
- A function is *surjective* if $f(X) = Y$, its range is the whole set Y, i.e. $\forall y \in Y \quad \exists x \in X; f(x) = y$
- A function is *bijective* if it is both injective and surjective
- If $f : X \to Y$ and $g : Y \to Z$, their *composition* $g \circ f$ is the function of X into Z, $g(f(x))$.

Functions II

For example, this function maps A to B, $f : A \to B$



Functions III

Class Exercise

What are the properties of $f(x) = x^2$ for $x \in \mathbb{R}$. How can you change its image and domain to make it (i) injective, (ii) surjective and (iii) bijective?

End of Theme 1

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