DIW GRADUATE CENTER

Metrics, Sequences and Limits

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Takeaways

- The real number set, $\mathbb{R}$ and its relation to Euclidean space.
- The concept of Euclidean distance.
- The concept of a metric and metric spaces.
- The concept of a vector norm and normed vector spaces.
- Open, closed sets and Lipschitz equivalence.
- Convergent sequences (in $\mathbb{R}$), Cauchy sequences and complete spaces.
- The contraction mapping theorem.
- Limits and continuity of functions.
- The extreme and intermediate value theorems.
- The concept of infinity.

Field I

Definition

A *field* $F = \{F, +, \cdot\}$ is an algebraic structure formed by a set F together with the two operations addition and multiplication, $(+, \cdot)$. These have the following properties:

1. Addition: $\forall \alpha, \beta, \gamma \in F$ the properties hold:
    1. Associative: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$
    2. Commutative: $\alpha + \beta = \beta + \alpha$
    3. Existence of a unique additive identity: $\exists ! 0 \in F : \alpha + 0 = 0 + \alpha = \alpha \quad \forall \alpha \in F$
    4. Existence of a unique additive inverse: $\forall \alpha \in F \quad \exists !(-\alpha) \in F : \alpha + (-\alpha) = (-\alpha) + \alpha = 0.$

Class Exercise

Which of these properties are/are not satisfied by the set of natural numbers, $\mathbb{N}$?

Field II

Definition

2. Multiplication: $\forall \alpha, \beta, \gamma \in F$ the following properties hold:
   1. Associative: $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$
   2. Commutative: $\alpha \cdot \beta = \beta \cdot \alpha$
   3. Existence of a unique multiplicative identity: $\exists ! 1 \in F : \alpha \cdot 1 = 1 \cdot \alpha = \alpha \quad \forall \alpha \in F$
   4. Existence of a unique multiplicative inverse: $\forall \alpha (\neq 0) \in F, \exists ! \alpha^{-1} \in F : \alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1.$
3. Multiplication is distributive w.r.t. addition: $\forall \alpha, \beta, \gamma \in F, \alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$

Class Exercise

Which of these properties are/are not satisfied by the set of integers, $\mathbb{Z}$? (You can use proof by contradiction)

The Real Number Set, $\mathbb{R}$ I

Field Axiom

The set $\mathbb{R}$ is endowed with the above mentioned field properties.

Order Axiom

There exists a complete ordering defined on $\mathbb{R}$ that is compatible with addition and multiplication in the following sense:

$$\forall \alpha, \beta, \gamma \in \mathbb{R}, \alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$$

and for

$$\alpha \leq \beta \quad \text{and} \quad 0 \leq \gamma \Rightarrow \alpha\gamma \leq \beta\gamma$$

The Real Number Set, $\mathbb{R}$ II

Completeness Axiom

Let L and H be nonempty sets of real numbers with the property that

$$\forall l \in L \quad \text{and} \quad \forall h \in H, l \leq h.$$

Then $\exists \alpha \in \mathbb{R}$ s.t.

$$\forall l \in L \quad \text{and} \quad \forall h \in H, l \leq \alpha \leq h.$$

Question

The union of which two sets make up $\mathbb{R}$?

Cartesian Product

Definition

Given two sets X and Y, their *Cartesian product* $X \times Y$ is the set of all ordered pairs (*tuples*) formed by an element of X followed by an element of Y. Hence,

$$X \times Y = \{(x, y); x \in X, y \in Y\}.$$

For the field of real numbers, $\mathbb{R}$, we denote its n-th dimension as

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}.$$

An element $\mathbf{x} \in \mathbb{R}^n$ is $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Note

Bold is used to denote vectors and matrices by convention.

Vector Space

Definition

Given a field, F, a *vector space* over F is a set V with two operations, addition and scalar multiplication, such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c, d \in F$ the following rules are satisfied:

1. *Addition is closed*: $\mathbf{u} + \mathbf{v} \in V$
2. *Scalar multiplication is closed*: $c\mathbf{v} \in V$
3. *Addition is associative*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. *Addition is commutative*: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
5. *Additive identity*: $\exists\ \mathbf{0} \in V$ s.t. $\mathbf{v} + \mathbf{0} = \mathbf{v}$
6. *Additive inverse*: $\exists -\mathbf{v} \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
7. *Associative scalar multiplication*: $(cd)\mathbf{v} = c(d\mathbf{v})$
8. *Multiplicative identity*: For $1 \in F$, $1\mathbf{v} = \mathbf{v}$
9. *Distributivity I*: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
10. *Distributivity II*: $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$.

Euclidean Space

Definition

$\mathbb{R}^n$ is also referred to as a finite dimensional *Euclidean space*.

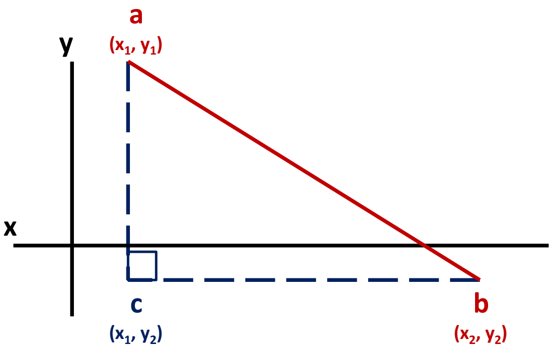
Class Exercise

Show that the Euclidean space $\mathbb{R}^2$ is a vector space.

Euclidean Distance I

The distance between two points, say **a** and **b** in $\mathbb{R}^2$ can be derived from the Pythagorean Theorem as

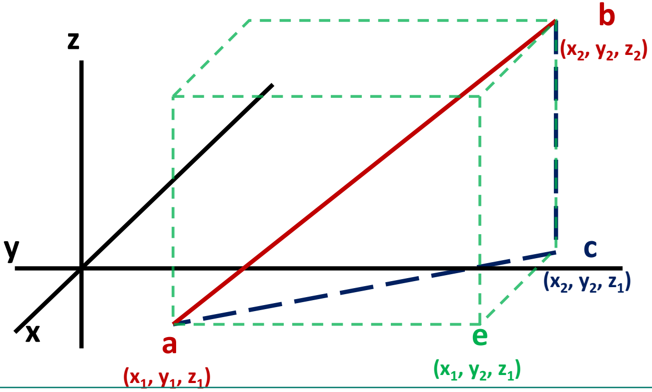
$$d_E(\mathbf{a}, \mathbf{b}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$



Euclidean Distance II

This result readily generalizes to two points **a** and **b** in $\mathbb{R}^3$ as

$$d_E(\mathbf{a}, \mathbf{b}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$



Euclidean Distance III

Following this line of reasoning, the distance between two points **a** and **b** in $\mathbb{R}^n$ is given by

$$d_E(\mathbf{a}, \mathbf{b}) = |\mathbf{a} - \mathbf{b}|_E = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}.$$

This is usually referred to as *Euclidean distance*.

Metric Space I

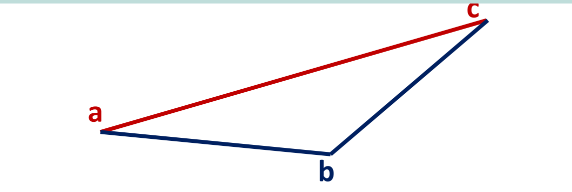
Definition

A *metric space* is a pair (X, d), where X is a set and d a metric defined on it. A *metric* is a real-valued nonegative function $d : X \times X \to \mathbb{R}$ describing the distance between neighboring points in a given set. For $a, b, c \in X$, a metric satisfies:

1. $d(a, b) > 0$ if $a \neq b$ and $d(a, b) = 0$ when $a = b$.
2. $d(a, b) = d(b, a)$ symmetry.
3. $d(a, c) \leq d(a, b) + d(b, c)$ triangle inequality.

Metric Space II

Figure: Triangle Inequality



Class Exercise

Show that the following are metric spaces

1. The set $\mathbb{Z}$ with $d_A(a, b) = |a - b|$ (absolute value metric).
2. The set $\mathbb{Z}$ with $d_T(a, b) = 0$ if $a = b$ and $d(a, b) = 1$ if $a \neq b$ (trivial metric).

Normed Vector Space

Definition

Given a vector space X, a *vector norm*, $\|\cdot\|$ is a real-valued function, $\|\cdot\| : X \to \mathbb{R}$, which measures the distance between $\mathbf{a} \in X$ and $\mathbf{0}$ (the origin).

Definition

A *normed vector space* is a vector space X, equipped with a vector norm, such that $\forall\, \mathbf{a}, \mathbf{b} \in X$ and any scalar k:

1. $\|\mathbf{a}\| > 0$, nonegativity
2. $\|\mathbf{a}\| = \mathbf{0} \Leftrightarrow \mathbf{a} = \mathbf{0}$, only the null vector has norm 0
3. $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$, triangle inequality
4. $\|k\mathbf{a}\| = |k| \cdot \|\mathbf{a}\|$.

Vector Norm: Examples I

p-norm

The *p-norm* of the n dimensional vector $\mathbf{a} \in X$ is given as

$$\|\mathbf{a}\|_p = \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p}.$$

$L - 2$-norm

For $\mathbf{a} \in \mathbb{R}^n$, the *L-2 norm* is

$$\|\mathbf{a}\|_2 = \sqrt{\sum_{i=1}^{n} (a_i)^2}.$$

Note that the absolute value is dropped for the squares of real numbers.

Vector Norm: Examples II

sup-norm

The *sup norm* for $\mathbf{a} \in \mathbb{R}^n$ is

$$\|\mathbf{a}\|_S = \sup_i \{|a_i|; i = 1, \ldots, n\}$$

i.e. it is the absolute value of its largest component.

Class Exercise

Show that the following are normed vector spaces

1. The set $\mathbb{R}^n$ with $\|\mathbf{a}\|_2 = (\sum_{i=1}^n a_i^2)^{1/2}$ ($L-2$ norm). **Hint**: Use the Cauchy-Schwartz inequality: Given two vectors, **x** and **y**, $\sum_{i=1}^n x_i y_i \leq \sqrt{\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2}$.
2. The set $\mathbb{R}^n$ with $\|\mathbf{a}\|_1 = \sum_{i=1}^n |a_i|$ ($L-1$ norm).
3. The set $\mathbb{R}^n$ with $\|\mathbf{a}\|_S = \sup_i |a_i|$ (*sup* norm).

Metrics and Vector Norms

A metric can be written as a vector norm as

$$d(a, b) = \|a - b\|.$$

Note for $\mathbf{a} \in \mathbb{R}^n$, the $L - 2$ norm and the Euclidean distance metric are related as

$$\|\mathbf{a}\|_2 = d_E(\mathbf{a}, \mathbf{0}).$$

Hence, any normed vector space can be viewed as a metric space. *However, not visa versa!* Why?

Open, Closed Balls and Bounded Sets

Definition

Given a metric space (X, d), (where $X \subseteq \mathbb{R}^n$), the *open ball* with center $\mathbf{a} \in X$ and radius ε is the set

$$B_\varepsilon^o(\mathbf{a}) = \{\mathbf{b} \in X; d(\mathbf{a}, \mathbf{b}) < \varepsilon\}.$$

Definition

A *closed ball* is defined analogously

$$B_\varepsilon^c(\mathbf{a}) = \{\mathbf{b} \in X; d(\mathbf{a}, \mathbf{b}) \leq \varepsilon\}.$$

Definition

A set is *bounded* if there exists a closed ball of finite radius that contains it. In other words, $d(\mathbf{a}, \mathbf{b}) \leq k \quad \forall \quad \mathbf{a}, \mathbf{b} \in X$ and some constant $k < \infty$.

Lipschitz Equivalence I

Definition

Two norms $\|\cdot\|_x$ and $\|\cdot\|_y$ are *Lipschitz-equivalent* if $\exists\ m, M \in \mathbb{R}$ where $m, M > 0$ s.t. for any vector **a**

$$m\|\mathbf{a}\|_x \leq \|\mathbf{a}\|_y \leq M\|\mathbf{a}\|_x.$$

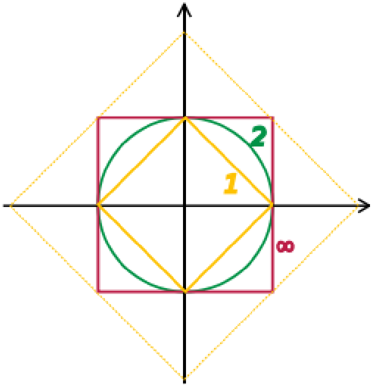
Example

In $\mathbb{R}^2$ graph the closed balls $B_1^c(\mathbf{0})$ for the $L-1, L-2$ and $L-\infty$ norms.

Example

From the exercise above, show that the $L-p$ norms, for $p = 1, 2, \ldots, \infty$ are Lipschitz equivalent.

Lipschitz Equivalence II



Theorem

All norms in $\mathbb{R}^n$ are Lipschitz equivalent.

Open and Closed Sets

Definition

Given a metric space (X, d), a set A in X is *open* if

$$\forall\, a \in A, \exists\, B^o_\varepsilon(a) \subseteq A.$$

Definition

Given a metric space (X, d), a set A in X is *closed* if its complement is open.

- $\emptyset$ and U are both open and closed in U
- The union of open (closed) sets is open (closed)
- The intersection of open (closed) sets is open (closed).

Convergent Sequences I

Definition

Let (X, d) be a metric space and $\{x_n\}_{n=0}^{\infty}$ a sequence in X, we say that $\{x_n\}_{n=0}^{\infty}$ converges to $x \in X$ if

$$\forall\, \varepsilon > 0, \;\; \exists\, N(\varepsilon) \in \mathbb{N}; d(x_n, x) < \varepsilon \text{ whenever } n \geq N(\varepsilon).$$

This is written as, $x_n \in B^o_\varepsilon(x)$ or $\lim_{n\to\infty} x_n = x$ or $x_n \to x$

Example

Show that $\lim_{n\to\infty} \frac{1}{n} = 0$ for $n \in \mathbb{N}$. **Hint**: Choose $N(\varepsilon) = [1/\varepsilon]_+$, the smallest integer greater than $1/\varepsilon$.

Class Exercise

Show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ for $n \in \mathbb{N}$.

Convergent Sequences II

Theorem

The limit of a convergent sequence is unique.

Example

Show that if $x_n \to x$ and $x_n \to y$, then $x = y$ (uniqueness of the limit). **Hint**: Use proof by contradiction and the triangular inequality. Assume sequences in $\mathbb{R}$.

Theorem

Every convergent sequence is bounded in the sense that $\exists\ M < \infty$ s.t. $|x_n| \leq M \quad \forall\ n \in \mathbb{N}$.

Class Exercise

Is the sequence $\{x_n\}_{n=0}^{\infty} = \{(-1)^n\}$ bounded and convergent?

Sequences in $\mathbb{R}$ I

Definition

If X is a set of real numbers bounded from above, then its smallest upper bound s is called the *supremum* and satisfies

1. $\forall\, x \in X, x \leq s$.
2. $\forall\, y < s, \exists\, x \in X$ s.t. $x > y$.

If $s \in X$, then it is called a *maximum*.

Definition

When X is bounded from below its largest lower bound or *infimum (minimum)* is defined analogously.

Definition

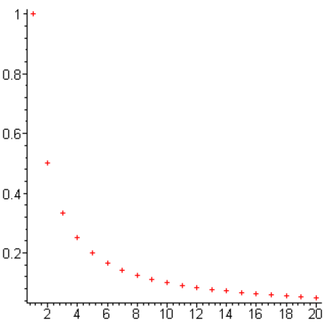
A *monotone* sequence is a sequence of real numbers that is either increasing if $x_{n+1} \geq x_n$ or decreasing if $x_{n+1} \leq x_n, \forall\, n$.

Sequences in $\mathbb{R}$ II

Theorem

Every bounded monotone sequence is convergent.

Figure: $\{x_n\}_{n\in\mathbb{N}} = \{1/n\}$



Sequences in $\mathbb{R}$ III

Theorem

Suppose the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ converge to x and y respectively. If $x_n \leq y_n$ or $x_n < y_n \; \forall \; n$, then $x \leq y$.

Example

To illustrate why we have the $\leq$ sign in the above theorem take for instance, $\{x_n\}_{n=0}^{\infty} = \{-(1/n)\}$ and $\{y_n\}_{n=0}^{\infty} = \{1/n\}$. Then, $x_n < y_n \; \forall \; n$, but in the limit $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

Sequences in $\mathbb{R}$ IV

Exponential Limit

For $\{x_n\}_{n=0}^{\infty} = \{(1 + (1/n))^n\}$ the limit is

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

This is known as the *exponential limit* (see Calculus section).

Theorem

For any $\alpha \in \mathbb{R}$ we have the following:

- If $|\alpha| < 1$, then $\lim_{n\to\infty} \alpha^n = 0$
- If $|\alpha| > 1$, then $\lim_{n\to\infty} \alpha^n = \infty$.

Sequences in $\mathbb{R}$ V

Theorem

Suppose the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ converge to x and y respectively. Then:

1. $\lim_{n\to\infty}(x_n \pm y_n) = x \pm y$
2. $\lim_{n\to\infty}(x_n y_n) = xy$
3. $\lim_{n\to\infty}(x_n/y_n) = x/y$, provided that $y_n \neq 0 \; \forall \; n$ and that $y \neq 0$.

Class Exercise

Show that $\lim_{n\to\infty} \frac{n^2+12n+4}{3n^2+5n+1} = 1/3$.

Subsequences in $\mathbb{R}$ I

Theorem

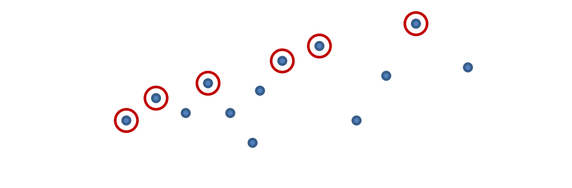
Every sequence of real numbers contains a monotone subsequence.

Theorem

Every bounded real sequence contains at least one convergent subsequence. *Bolzano-Weierstrass.*

Subsequences in $\mathbb{R}$ II

Figure: Monotone Subsequence



Class Exercise

Show that the sequence $\{x_n\}_{n=0}^{\infty} = \{(-1)^n\}$ contains at least one monotone convergent subsequence.

Cauchy Sequences I

It may not be feasible to calculate the limit of a sequence. In that case we can make use of the following sequence

Definition

A sequence in a metric space, (X, d) is *Cauchy* if

$$\forall\, \varepsilon > 0, \quad \exists\, N(\varepsilon) \in \mathbb{N}; \quad d(x_m, x_n) < \varepsilon\, \forall\, m, n \geq N(\varepsilon).$$

Theorem

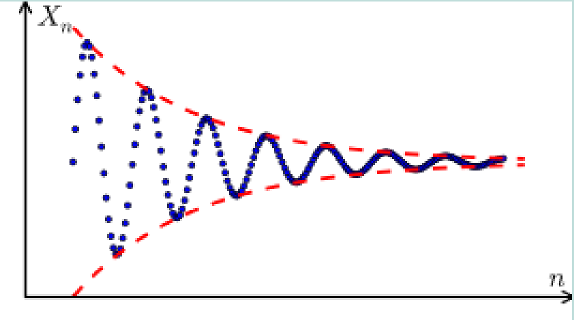
Every convergent sequence in a metric space is Cauchy.

Example

Prove the above theorem. **Hint**: Use the triangular inequality.

Cauchy Sequences II

Figure: Cauchy Sequence



Advantage

We just need to know the terms of the sequence rather than a specific limit, x.

Complete Spaces

Definition

A metric space, (X, d) is *complete* if every Cauchy sequence contained in X converges to some $x \in X$. A complete normed vector space is called a *Banach space*.

Theorem

Any finite dimensional Euclidean space $E^m = (\mathbb{R}^m, d_E)$ is complete.

Class Exercise

1. Show that the set $\mathbb{Z}$ with $d(a, b)_A = |a - b|$ is complete.
2. Give an example to show that the set of rational numbers, $\mathbb{Q}$ is not complete.

Contraction Mapping Theorem I

Definition

Assume $T : X \to X$ is an operator on the metric space, (X, d). Then T is a *contraction of modulus* β if for some $\beta \in (0, 1)$ we have:

$$\forall a, b \in X, d(Ta, Tb) \leq \beta d(a, b)$$

Informally, a contraction brings any two points of a set closer to each other.

It has uniformly a slope less than 1 in absolute value.

Contraction Mapping Theorem II

Theorem

Let (X, d) be a complete metric space and $T : X \to X$ a contraction with modulus β, then:

1. T has exactly one fixed point a^* in X such that $Ta^* = a^*$
2. The sequence

$$a_1 = Ta_0, a_2 = Ta_1, \ldots, a_{n+1} = Ta_n$$

converges to a^* for any starting point $a_0 \in X$.

Contraction Mapping Theorem III

- Can be used to prove the existence and uniqueness of solutions to problems related to dynamic optimization
- Beginning with a trail solution we can always approximate the true solution arbitrarily closely
- **Blackwell's theorem** gives sufficient conditions on whether an operator in a function space is a contraction. This usually needs to be checked in practice.
- Side note: Other fixed point theorems you may encounter in you classes are *Brouwer's, Kakutani's* and *Tarsky's* fixed point theorems (and others).

Limits of Functions I

Definition I

Let (X, d) and (Y, ρ) be two metric spaces with a function $f : X \to Y$ and let x^0 be a limit point of X. Then f has a *limit* L as x approaches x^0 if

$$\forall\, \varepsilon > 0 \;\exists\, \delta(\varepsilon) > 0;\;\; \rho(f(x), L) < \varepsilon \text{ whenever } 0 < d(x, x^0) < \delta(\varepsilon).$$

We write $f(x) \to L$ as $x \to x^0$ or $\lim_{x \to x^0} f(x) = L$.

Note

In Euclidean space this definition is equivalent to

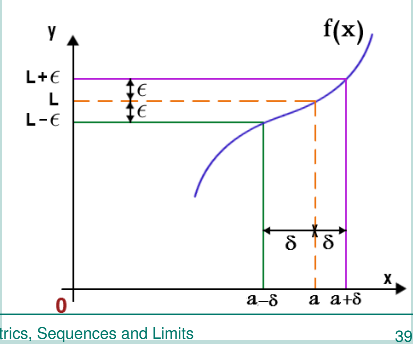
$$\forall\, \varepsilon > 0 \;\exists\, \delta(\varepsilon) > 0;\;\; |f(x) - L| < \varepsilon \text{ whenever } 0 < |x - x^0| < \delta(\varepsilon).$$

Limits of Functions II

Note

The above is equivalent to checking whether the left and right side limits are equal at x^0.

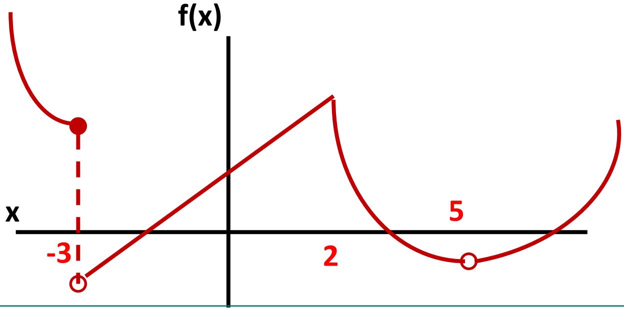
Figure: Limit of a Function: $\lim_{x\to a} f(x) = L$.



Limits of Functions III

Class Exercise

1. Does the limit of $f(x) = \frac{|x-3|}{x-3}$ at $x^0 = 3$ exist? **Hint**: first draw the graph.
2. Do the limits at $x^0 = -3, 2$ and 5 in the graph exist?



Limits of Functions IV

Theorem

Let (X, d) be a metric space and $(Y, \|\cdot\|)$ a normed vector space. Let f and g be functions mapping $X \to Y$ and let x^0 be a limit point of X. Further, assume that $f(x) \to a$ and $g(x) \to b$. Then

- $f(x) + g(x) \to a + b$ as $x \to x^0$
- for any scalar $\gamma, \gamma f(x) \to \gamma a$ as $x \to x^0$
- $f(x)g(x) \to ab$ as $x \to x^0$ if $(Y, \|\cdot\|)$ is $\mathbb{R}$
- $f(x)/g(x) \to a/b$ as $x \to x^0$ if $(Y, \|\cdot\|)$ is $\mathbb{R}$ and provided that $b \neq 0$.

Continuity of Functions I

Epsilon-Delta Definition

Let (X, d) and (Y, ρ) be two metric spaces with a function $f : X \to Y$. f is *continuous* at a point $x^0 \in X$ if

$$\forall\, \varepsilon > 0,\ \ \exists\, \delta(\varepsilon, x^0) > 0 \qquad ; \rho(f(x), f(x^0)) < \varepsilon$$
$$\text{whenever} \qquad d(x, x^0) < \delta(\varepsilon, x^0).$$

Again in Euclidean space the metrics d and ρ are absolute value metrics.

Class Exercise

Show that the following functions are continuous

1. $f(x) = x^2$ for all $x^0 \in \mathbb{R}$. **Hint**: choose $\delta(\varepsilon, x^0) = \min\{1, \varepsilon/(2|x^0| + 1)\}$.
2. $f(x) = 2x$ for all $x^0 \in \mathbb{R}$
3. $f(x) = 1/x$ for all $x^0 \in \mathbb{R}^+ \setminus \{0\}$ (optional).

Continuity of Functions II

Limit Definition

Let f be a function $f : X \to Y$. f is *continuous* at a point $x^0 \in X$ if

- $\lim_{x \to x^0} f(x)$ exists
- $f(x^0)$ is defined and
- $\lim_{x \to x^0} f(x) = f(x^0)$.

Class Exercise

Verify if the function in the figure above is continuous at the specific points.

Note, limits and continuity of multivariate functions are discussed in the Calculus section.

Continuity of Functions III

Sequence Definition

Let (X, d) and (Y, ρ) be two metric spaces with a function $f : X \to Y$. f is *continuous* at a point $x^0 \in X$ if for every sequence $\{x_n\}$ convergent to x^0 in (X, d), the sequence $\{f(x_n)\}$ converges to $f(x^0)$ in (Y, ρ).

Extreme Value Theorem I

Definition

A subset X of $\mathbb{R}^n$ is said to be *compact* if it is closed and bounded.

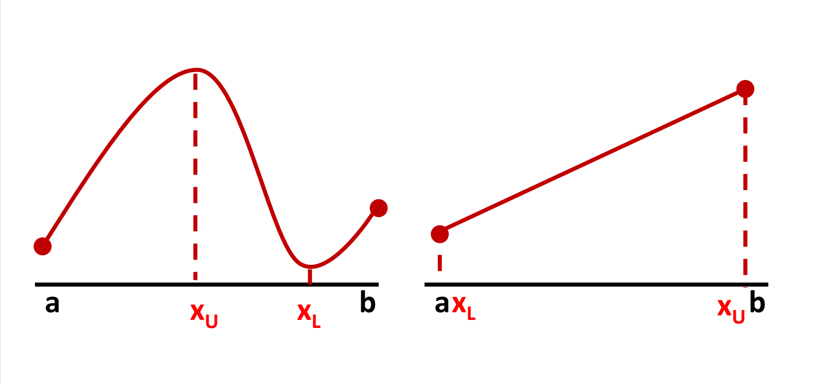
Extreme Value Theorem

Let f be a continuous real-valued function on the compact interval $[a, b]$. Then there exist points x_L and x_U in the interval s.t. $\forall\ x \in [a, b], f(x_L) \leq f(x) \leq f(x_U)$.

Extreme Value Theorem II

Both the function's maximum and minimum are attained on the interval.

Figure: EVT

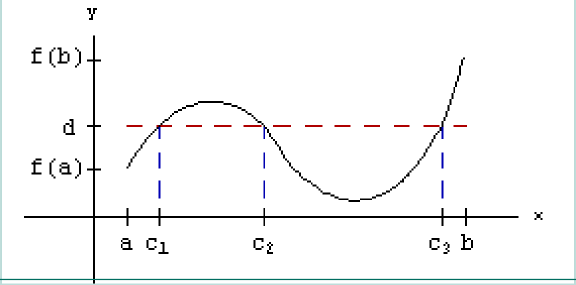


Intermediate Value Theorem

Intermediate Value Theorem

Let f be a continuous real-valued function on the compact interval $[a, b]$. Then for each number d strictly between $f(a)$ and $f(b)$ there exists a point $c \in (a, b)$ s.t. $f(c) = d$.

Figure: IVT



Monotonic Functions

Theorem

Let f be a monotonically increasing function on (a, b). Then the one-sided limits

$$f(x^-) = \lim_{y \to x^-} f(y) \quad \text{and} \quad f(x^+) = \lim_{y \to x^+} f(y)$$

exist at every point x on (a, b) and

$$\sup\{f(s); a < s < x\} = f(x^-) \leq f(x) \leq f(x^+) = \inf\{f(s); x < s < b\}.$$

Infinite Limits

The proceeding results generalize to infinite limits, but caution should be taken. For instance:

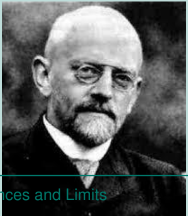
- If $\lim_{x \to x^0} f(x) = y^0$ and If $\lim_{x \to x^0} g(x) = \infty$, then $\lim_{x \to x^0}(f(x) + g(x)) = \infty$, **but** if $\lim_{x \to x^0} f(x) = -\infty$, $\lim_{x \to x^0}(f(x) + g(x))$ can be anything.
- If $\lim_{x \to x^0} f(x) = y^0 > 0$ and $\lim_{x \to x^0} g(x) = \infty$, then $\lim_{x \to x^0}(f(x)g(x)) = \infty$, **but** if $\lim_{x \to x^0} f(x) = 0$ nothing can be said about $\lim_{x \to x^0}(f(x)g(x))$ without further study.
- If $\lim_{x \to x^0} f(x) = y^0 > 0$ and $\lim_{x \to x^0} g(x) = 0$, then $\lim_{x \to x^0} f(x)/g(x) = \infty$.

In some of these situations it is useful to use L' Hôpital's rule (see section on Calculus).

But What is ∞? Hilbert's Infinite Hotel Paradox I

- We often use the symbol ∞ in mathematics, but its interpretation is not trivial.
- Consider Hilbert's infinite hotel paradox: *Suppose we have a hotel with infinitely many rooms which are all full.*
  - Would it be possible for you to get a room?
  - Would it be possible for you and a friend to get a room?
  - Would it be possible for infinitely many people to get a room?

David Hilbert



But What is ∞? Hilbert's Infinite Hotel Paradox II

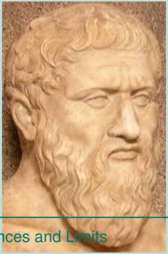
- How many rooms would be occupied if the next day everyone checked out?
- How many rooms would be occupied if the next day everyone but you checked out?
- How many rooms would be occupied if the next day everyone but you and your friend checked out?

Georg Cantor

Zeno's Paradoxes

- Apparently the infinite seems to create some impossible situations.
- Ancient Greek philosopher Zeno formulated several of these paradoxes.
- The next slides are based on Brain Dunning's Skeptoid podcast episode 267 at `www.skeptoid.com`

Zeno



Zeno's Paradoxes: *Homer and the bus stop*

Homer walks to the bus stop, but once he is half way there he must walk half the remaining distance and then half of that distance and so on. In other words once he has 1/2 the distance remaining he then has 1/4 and then 1/8 and so on. With an infinite amount of distances to travel he can therefore never reach the bus stop.

Homer

Zeno's Paradoxes: *Achilles and the turtle*

Achilles and a turtle have a foot race. Knowing he is the faster runner, Achilles gives the turtle a 100m head start. In the time it takes Achilles to run the 100m the turtle moves 10m ahead so that it is still ahead of Achilles. In the time it takes Achilles to run the extra 10m the turtle moves a further 1m ahead. No matter how many times Achilles advances to the turtle's last position, the turtle always moves a little bit ahead. It is impossible for Achilles to catch up to the turtle.

Zeno's Paradoxes: *The fletcher's arrows*

A fletcher finds that all his arrows are unable to move. Time consists of an infinite succession of moments in each of which the arrow is frozen in flight since it does not have time in any individual moment for it to move. It is impossible for the fletcher to shoot a single arrow.

Zeno's Paradoxes and Proposed Solutions I

- A proposed solution to the first and second paradox seems to come from quantum theory.
- It holds that the smallest possible unit is a Planck length, which is $1.62 \times 10^{-35} m$.
- This would make the distance lengths finite and therefore Homer would be able to reach the bus stop and Achilles would be able to catch up to the turtle.
- However, this is a geometric problem and a quantum solution would not work.
- Imagine a right triangle with sides of 1 Planck length, its hypothenuse would have to be $\sqrt{2}$ Planck lengths which is not possible.

Zeno's Paradoxes and Proposed Solutions II

We are in fact *NOT* dealing with infinities!

Fact

$0.99999.... = 1$

Proof

Simplest way is to divide both by 3 and you get 0.33333....,
hence, they are equivalent!

- The fletcher's paradox ignores that speed is a function of distance and time. Put simply you can never make a photo of a moving object without some blur. An object is *NOT* frozen in a given instant of time.

Zeno's Paradoxes and Proposed Solutions III

- The second paradox also omits time. The physical length of each segment it takes Achilles to come closer to the turtle decreases exponentially in a converging sequence and so does the time it takes for Achilles to cross it. Achilles *will* catch up to the turtle.
- Homer does not make a journey of infinite length. In fact his journey can be summarized by the following equation

$$\sum_{n=1}^{\infty}(1/2)^n,$$

which is an absolutely converging sequence.

Individual Exercise

What does it equal?

So Homer *does* reach the bus stop.

End of Theme 2

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