

# Concavity

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# Takeaways

- Convex sets
- Hyperplanes and hyperplane theorems
- Concave and quasiconcave functions
- Correspondences and hemicontinuity.

# Convex Sets

## Definition

The set  $A \subset \mathbb{R}^n$  is *convex* if for  $x, y \in A$  we have that

$$\lambda x + (1 - \lambda)y \in A$$

for all  $\lambda \in [0, 1]$ .

*No line segment connecting any two points within the set can go outside the set.*

## Class Exercise

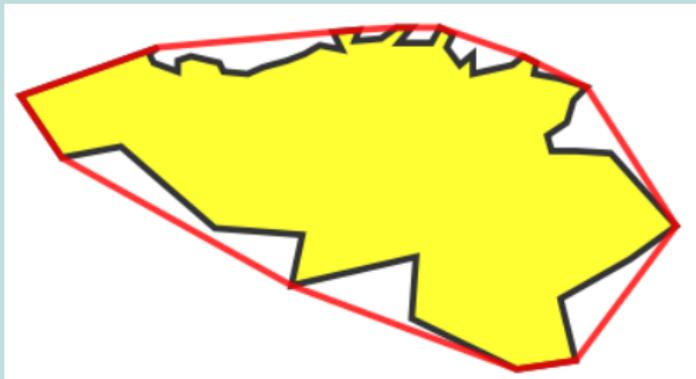
Show by means of a Venn diagram that the intersection of two convex sets is convex, while the union of two convex sets is not necessarily convex.

# Convex Hull

## Definition

For the set  $A \subset \mathbb{R}^n$ , the *convex hull* of  $A$ , denoted  $Co(A)$ , is the smallest convex set containing  $A$ . In other words the intersection of all convex sets containing  $A$ .

## Figure: Convex Hull



# Hyperplane I

## Definition

Let  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  be scalars such that not all of them are zero. Let  $\mathbf{w} \in \mathbb{R}^n$  such that

$$\mathbf{a}'\mathbf{w} = c$$

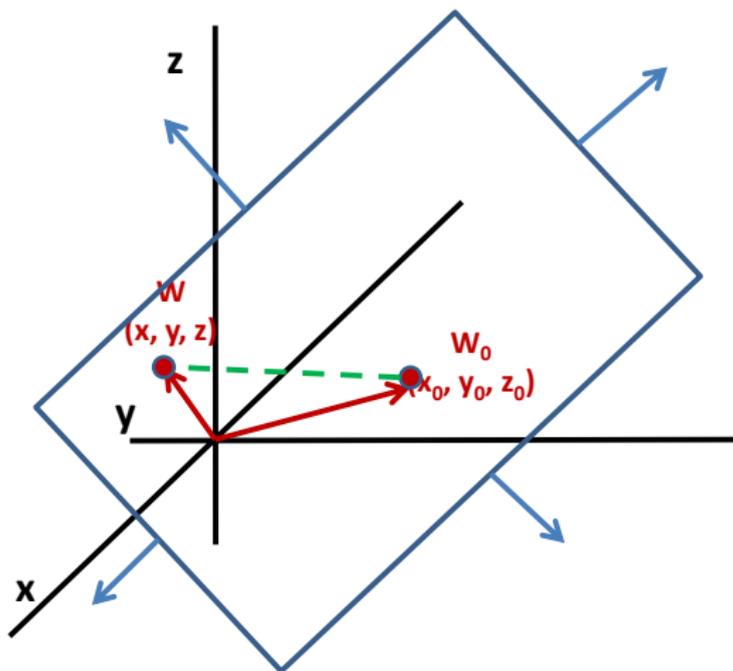
for some constant  $c \in \mathbb{R}$ . This is then known as the *hyperplane*,  $H_{\mathbf{a},c}$  generated by  $\mathbf{a}$  and  $c$ .

*A hyperplane in  $\mathbb{R}^2$  is a line, in  $\mathbb{R}^3$  it is a surface.*

## Class Exercise

A plane in  $\mathbb{R}^3$  is also defined as  $Ax + By + Cz = D$ . Show that this is equivalent with the definition above.

# Hyperplane II



# Half-Space

## Definition

The *half-space above* and the *half-space below* the hyperplane  $H_{\mathbf{a},c}$  is given by the sets  $\{\mathbf{w} \in \mathbb{R}^n : \mathbf{a}'\mathbf{w} \geq c\}$  and  $\{\mathbf{w} \in \mathbb{R}^n : \mathbf{a}'\mathbf{w} \leq c\}$  respectively.

# Hyperplane Theorems

## Separating Hyperplane Theorem

Suppose that  $B \subset \mathbb{R}^n$  is a convex and closed set and that  $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \notin B$ . Then there is a non zero vector  $\mathbf{a} \in \mathbb{R}^n$  and a point  $c \in \mathbb{R}$  s.t.  $\mathbf{a}'\mathbf{x} > c$  and  $\mathbf{a}'\mathbf{y} < c \quad \forall \mathbf{y} \in B$ .

## Supporting Hyperplane Theorem

Roughly speaking, the *supporting hyperplane theorem* uses the weak inequality sign.

## Intuitively

The hyperplane theorems set out the conditions to guarantee that there is a hyperplane separating (supporting) disjoint sets in  $\mathbb{R}^n$ . These theorems are used to prove for instance the second welfare theorem.

# Concave Functions I

## Definition I

The function  $f : A \rightarrow \mathbb{R}$ , where  $A$  is a convex set  $A \subseteq \mathbb{R}^n$ , is *concave* if for  $\mathbf{x}, \mathbf{y} \in A$  we have that

$$f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \geq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y})$$

for all  $\lambda \in [0, 1]$ . A *strictly concave* function has the same definition with a strict inequality sign.

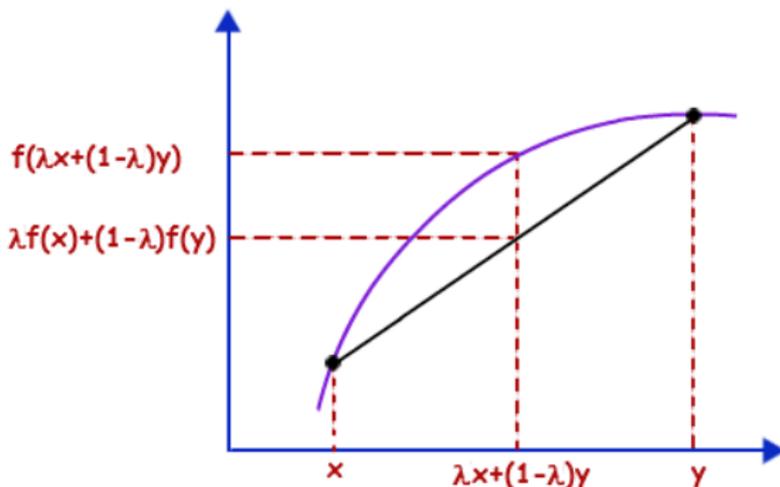
## Convex Function

Reversing the inequality sign in the above definition defines a *convex* function.

## Note

A convex set in  $\mathbb{R}$  is a closed interval.

# Concave Functions II



## Class Exercise

Is a linear function of the form  $f(x) = ax$  for some constant  $a$  a concave, a convex or some other type of function?

# Concave Functions III

## Definition II

A continuously differentiable function  $f : X \rightarrow \mathbb{R}$  is *concave* iff

$$f(x + y) \leq f(x) + \nabla_x \cdot y$$

$\forall x \in X$  and  $y \in \mathbb{R}^n$  (with  $x + y \in X$ ).

## Class Exercise

Illustrate the above definition by means of a concave function in  $\mathbb{R}^2$ .

# Concave Functions III

## Theorem

A twice continuously differentiable function  $f : X \rightarrow \mathbb{R}$  is *concave* iff the Hessian,  $\mathbf{H}$  is negative semi definite for all  $x \in X$ .

## Question

How does this translate to a function with a single variable?

# Quasiconcave Functions I

## Definition I

Let  $x, y$  be any two distinct points of a function  $f : X \rightarrow \mathbb{R}$ , where  $X$  is a convex set  $X \subset \mathbb{R}^n$ , then, for  $\lambda \in (0, 1)$ , we say a function is *quasiconcave* iff

$$f(y) \geq f(x) \Rightarrow f(\lambda x + (1 - \lambda)y) \geq f(x).$$

Similarly, a function is *quasiconvex* iff

$$f(y) \geq f(x) \Rightarrow f(\lambda x + (1 - \lambda)y) \leq f(y).$$

Note that this needs to hold for *all* possible points in  $X$ .

## Class Exercise

Using this definition, determine what the following functions are.

# Quasiconcave Functions II

## Definition II

A function  $f : X \rightarrow \mathbb{R}$ , where  $X$  is a convex set  $X \subset \mathbb{R}^n$  is quasiconcave (quasiconvex) iff, for any constant  $k$  the contour set

$$S^{\geq} := \{x; f(x) \geq k\} \quad (S^{\leq} := \{x; f(x) \leq k\})$$

is convex. Strict inequalities define strictly quasiconcave (quasiconvex) functions.

## Class Exercise

Use the examples in the previous exercise to verify this definition. Illustrate the contour set of a function in  $\mathbb{R}^3$ .

## Class Exercise

Check whether  $f(x) = x^2$  for  $x \geq 0$  is quasiconcave, quasiconvex. How about for  $x = (-\infty, \infty)$ ?

# Quasiconcavity and Concavity

## Theorem

A (strictly) concave (convex) function is (strictly) quasiconcave (quasiconvex).

## Note

This implies that quasiconcavity (quasiconvexity) is a weaker condition than concavity (convexity).

## Class Exercise

Prove the above theorem.

# Correspondence

Sometimes in economics we encounter the following

## Definition

Given a set  $A \subset \mathbb{R}^n$  and a set  $Y \subset \mathbb{R}^k$ , a *correspondence*  $f : A \rightarrow Y$  is a rule that assigns a set  $f(\mathbf{x}) \subset Y$  for every  $\mathbf{x} \in A$

- *The range of a correspondence is over sets instead of elements*
- In game theory for example, a best response of a player may be a set of several strategies.
- In macro, dynamic programming problems may involve correspondences.

# Upper Hemicontinuity

## Definition, Mas-Colell et al. (1995)

Given  $A \subset \mathbb{R}^n$  and the closed set  $Y \subset \mathbb{R}^k$ , the correspondence  $f : A \rightarrow Y$  is *upper hemicontinuous* if for any two sequences  $\mathbf{x}_n \rightarrow \mathbf{x} \in A$  and  $\mathbf{y}_n \rightarrow \mathbf{y}$ , with  $\{\mathbf{x}_n\} \in A$  and  $\{\mathbf{y}_n\} \in f(\{\mathbf{x}_n\})$ , we have  $\mathbf{y} \in f(\mathbf{x})$  and the images of compact sets are bounded, that is for every compact set  $B \subset A$  the set  $f(B) = \{\mathbf{y} \in Y : \mathbf{y} \in f(\mathbf{x}) \text{ for some } \mathbf{x} \in B\}$  is bounded.

## Definition, Lucas et al. (1989)

Given  $A \subset \mathbb{R}^n$  and the closed set  $Y \subset \mathbb{R}^k$ , the correspondence  $f : A \rightarrow Y$  is *upper hemicontinuous* if  $f(\mathbf{x})$  is nonempty and if, for every two sequence  $\mathbf{x}_n \rightarrow \mathbf{x} \in A$  and  $\mathbf{y}_n \rightarrow \mathbf{y}$ , such that  $\mathbf{y}_n \in f(\mathbf{x}_n)$  for all  $n$ , there exists a convergent subsequence of  $\{\mathbf{y}_n\}$  whose limit point  $\mathbf{y}$  is in  $f(\mathbf{x})$ .

# Lower Hemicontinuity

## Definition, Mas-Colell et al. (1995)

Given  $A \subset \mathbb{R}^n$  and the closed set  $Y \subset \mathbb{R}^k$ , the correspondence  $f : A \rightarrow Y$  is *lower hemicontinuous* if for any sequence  $\mathbf{x}_n \rightarrow \mathbf{x} \in A$  with  $\{\mathbf{x}_n\} \in A$  and every  $\mathbf{y} \in f(\mathbf{x})$ , we can find a sequence  $\mathbf{y}_n \rightarrow \mathbf{y}$  and an integer  $N$  s.t.  $\{\mathbf{y}_n\} \in f(\{\mathbf{x}_n\})$  for  $n > N$ .

## Definition, Lucas et al. (1989)

Given  $A \subset \mathbb{R}^n$  and the closed set  $Y \subset \mathbb{R}^k$ , the correspondence  $f : A \rightarrow Y$  is *lower hemicontinuous* if  $f(\mathbf{x})$  is nonempty and if, for every  $\mathbf{y} \in f(\mathbf{x})$  and for any sequence  $\mathbf{x}_n \rightarrow \mathbf{x}$ , there exists an integer  $N \geq 1$  and a sequence  $\{\mathbf{y}_n\}_{n=N}^{\infty}$  such that  $\mathbf{y}_n \rightarrow \mathbf{y}$  and  $\mathbf{y}_n \in f(\mathbf{x}_n)$  for all  $n > N$ .

# Hemicontinuity I

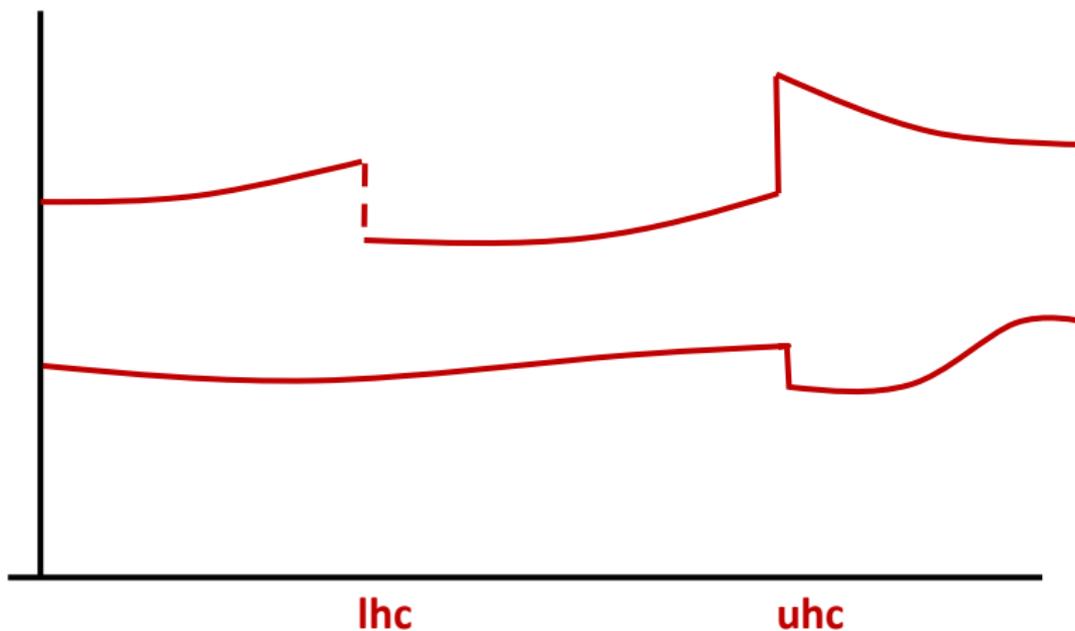
## Conclusion

Unclear what this means.

## Intuitively

Upper hemicontinuity is associated with 'explosions' of the set  $Y$ . The discontinuity is included in the set. A lower hemicontinuity is associated with 'implosions' of the set  $Y$ . The discontinuity is *not* included in the set.

# Hemicontinuity II



# End of Theme 6

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