Assignment 1

- The Classical Linear Regression Model -

Please hand in the exercises marked with a * by 9:00 a.m. on 29^{th} October.

In the following assume the classical linear regression model $y = X\beta + \epsilon$, where the vectors and matrices are defined as on the lecture slides.

Most exercises are based on the material in Hayashi (2000, Ch.1). Reading the chapter will be helpful for understanding the broader context. Optionally, for Problems 2 and 3, Chapter 2 in Davidson & MacKinnon (2004) also provides a good summary of the material covered.

Problem 1 * (4 Points)
Answer these two overview questions to the best of your knowledge.

(a) Why is the classical linear regression model so central to econometrics?

(b) What are some of the virtues and limitations of the least squares estimator?

Problem 2
Let $P_X = X(X'X)^{-1}X'$ and $M_X = I_T - P_X$. Let T stand for the number of observations and let e be the OLS residual.

(a) Show that $M_X X = 0$, $P_X M_X = M_X P_X = 0$, and $M_X \epsilon = M_X y = e$.

(b) Provide a geometric interpretation for P_X, M_X and the results from (a).

Hint: Reading Davidson & MacKinnon (2004, Ch. 2.1 - 2.4) may be of help.

Problem 3 * (4 points)
Let the sample of regressors be split into two parts $X = [X_1 \;\; X_2]$. Consider two model specifications:

(i) $y = X\beta + \epsilon = X_1\beta_1 + X_2\beta_2 + \epsilon$ is the CLRM with standard assumptions,

(ii) $y = M_2 X_1 \beta_1 + \nu$,

where $M_2 \equiv M_{X_2}$ is defined analogously to M_X in Problem 2.

(a) Show that $M_X M_2 = M_X$.

(b) Use the result from (a) to show that the OLS estimators for β_1 resulting from equation (i) and (ii) are numerically the same.

(c) Is it also the same as for $M_2 y = M_2 X_1 \beta_1 + u$? Provide an intuitive explanation for your answer. Note: The error term here is defined as $u = M_2 \epsilon$.

Hint: Reading Davidson & MacKinnon (2004, Ch. 2.1 - 2.4) may be of help.

Problem 4
Assume the classical linear regression model $y = X\beta + \epsilon$.

(a) Prove that $y'y = \hat{y}'\hat{y} + e'e$, where $\hat{y} = Xb$ and b is the OLS estimator.

(b) Assume that the regression has an intercept term. Derive the expression for the coefficient of determination

$$R^2 = 1 - \frac{e'e}{y'y - T\bar{y}^2} \ .$$

Where do you make use of the fact that the regression has an intercept?

(c) Prove that

$$R^2 = \frac{b'X'y - T\bar{y}^2}{y'y - T\bar{y}^2} = 1 - \frac{e'e}{y'y - T\bar{y}^2} \ .$$

d) Consider the model where $X = \mathbb{1}$ and $\mathbb{1} = [1 \cdots 1]'$. Show that in this model $R^2 = 0$.

Problem 5 * (11 points)
Assume $y_i, i = 1, \ldots, n$ to be realizations of i.i.d. exponentially distributed random variables $Y_i, i = 1, \ldots, n$ with parameter $\theta > 0$. The density of Y_i is:

$$f(y_i|\theta) = \begin{cases} 0, & \text{if } y_i < 0, \\ \frac{1}{\theta} exp(-\frac{y_i}{\theta}), & \text{if } y_i \geqslant 0 \end{cases}$$

Moreover, Y_i has expectation $\mathbb{E}[Y_i] = \mu = \theta$ and variance $\mathbb{V}[Y_i] = \sigma^2 = \theta^2$.

(a) Write down the log-likelihood $l(\theta|y)$.

(b) Derive the score function and compute its expectation.

(c) Find the maximum likelihood estimator $\hat{\theta}^{ML}$ for the parameter θ.

(d) Find the MLE for expectation and variance of Y_i.

Hint: Use the invariance property of the MLE.

(e) Show that $\hat{\theta}^{ML}$ is unbiased and compute its variance. Compute the expectation of the MLE for σ^2.

(f) Compute the Fisher Information matrix. Give the Cramér-Rao Lower Bound for θ. Does the variance of $\hat{\theta}^{ML}$ reach the Cramér-Rao Lower Bound for a finite sample size n?

Problem 6
Given the following results from an OLS regression with $T = 100$ observations

$$b = \begin{pmatrix} 0.2 \\ -0.9 \end{pmatrix}, \quad \hat{\sigma}^2 (X'X)^{-1} = \begin{pmatrix} 0.04 & 0.07 \\ 0.07 & 0.25 \end{pmatrix},$$

test the following null hypotheses:

(a) $H_0^I: \ \beta_1 = 0$.

(b) $H_0^{II}: \ \beta_1 = \beta_2$. Write H_0^{II} in the form $R\beta = r$.

(c) $H_0^{III}: \ \beta_1 + \beta_2 = -1$. Write H_0^{III} in the form $R\beta = r$.

Test H_0^{III}, using *both* the F-test and a simple t-test.

Problem 7
Now assume that, unlike in the classical linear model, you have

$$y = X\beta + \epsilon, \text{ with } \mathbb{E}[\epsilon\epsilon'] = \Omega \neq \sigma^2 I.$$

where Ω is known and positive definite. The $T \times K$ regressor matrix X is deterministic.

(a) Show that the estimators $\hat{\beta}_{OLS}$ and $\hat{\beta}_{GLS}$ are unbiased for β.

(b) Derive the covariance matrices of $\hat{\beta}_{OLS}$ and $\hat{\beta}_{GLS}$.

(c) Show that the GLS estimator is more efficient than OLS.

Hint: using $A = (X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}$ verify that the difference between both covariance matrices $\mathbb{V}[\hat{\beta}_{OLS}] - \mathbb{V}[\hat{\beta}_{GLS}]$ can be expressed as $A\Omega A'$ and show that $A\Omega A'$ is positive semidefinite.

Problem 8 * (6 Points)
This exercise builds on the empirical application of Hayashi (2000, pp. 60 - 69), which again builds on the analysis of the electricity industry in the US in 1955 by Nerlov (1963). Use the provided data set to complete the following tasks.

(a) Estimate the unrestricted model in equation (1.7.4) in Hayashi (2000, p. 63).

(b) Suppose that you wanted to test the null hypothesis that the coefficient on the price of fuel is really 0.4. Calculate the test statistic for this hypothesis and its p-value three different ways, and summarize in each case whether you would reject or fail to reject the hypothesis:

   a) A t-test of the null hypothesis that $\beta_{Pfuel} = 0.4$.

   b) Formulate an F test using equation (1.4.9). What are $\boldsymbol{R}$ and $\boldsymbol{r}$?

   c) Re-estimate the regression imposing the restriction and use formula (1.4.11).

(c) Suppose next that you wanted to test the null hypothesis of homogeneity $\beta_{Plabor} + \beta_{Pcapital} + \beta_{Pfuel} = 1$. Again test this restriction two ways, using formulas (1.4.9) and (1.4.11).

References

Davidson, R. & MacKinnon, J. G. (2004). *Econometric Theory and Methods*, volume 5. New York: Oxford University Press.

Hayashi, F. (2000). *Econometrics*. Princeton: Princeton University Press.

Nerlov, M. (1963). Returns to scale in electricity supply. In C. Christ (Ed.), *Measurement in Economics: Studies in Mathematical Economics and Econometrics in Memory of Yehuda Grunfeld* chapter 7, (pp. 167 – 198). Stanford: Stanford University Press.