Assignment 2

- Asymptotic Theory -

Please hand in the exercises marked with a * by 9:00 a.m. on 12th November.

In the following assume the classical linear regression model $y = X\beta + \epsilon$, where the vectors and matrices are defined as on the lecture slides.

Problem 1
Let $a_T = \sum_{t=1}^{T} t$. Show the following results.

(a) $a_T = O(T^2)$.

(b) $a_T = o(T^{5/2})$.

(c) $\frac{1}{a_T} = O(T^{-2})$.

Problem 2
Let $a_T = O_p(T)$, $b_T = o_p(T)$, $c_T = o_p(T)$, $d_T = O_p(T^2)$ be such that

$$A_T = \begin{bmatrix} a_T & b_T \\ c_T & d_T \end{bmatrix}$$

is invertible. Show that $\text{plim} A_T^{-1} = 0$.

Problem 3
Let ϵ_t be i.i.d.$(0, \sigma_\epsilon^2)$ with finite fourth moments and $x_t = O(1)$ a deterministic $(K \times 1)$ vector, $t = 1, 2, \dots$, such that $\lim\, T^{-1} \sum_{t=1}^{T} x_t x_t'$ is finite and positive definite. Show that

(a) $\sum_{t=1}^{T} x_t \epsilon_t = o_p(T)$.

(b) $T^{-1} \sum_{t=1}^{T} x_t \epsilon_t = o_p(1)$.

(c) $\text{plim}\; T^{-1} \sum_{t=1}^{T} x_t \epsilon_t = 0$.

(d) $\sum_{t=1}^{T} x_t \epsilon_t = O_p(T^{1/2})$.

Problem 4
Let ϵ_t be i.i.d.$(0, \sigma_\epsilon^2)$ and define $z_{h,t} = \epsilon_t \epsilon_{t-h}$.

(a) Show that $z_{h,t}$ is a martingale difference sequence for $h \neq 0$.

(b) Is $z_{h,t}$ also a martingale difference sequence for $h = 0$?

(c) Show that $T^{-1} \sum_{t=1}^{T} \epsilon_t \epsilon_{t-h} \xrightarrow{p} 0$ for $h \neq 0$.

Problem 5 * (5 Points)
Let $\{X_T\}$ be a sequence of Bernoulli random variables with

$$\begin{aligned} P(X_T = 0) &= 1 - 1/T, \\ P(X_T = 1) &= 1/T, \end{aligned}$$

and $T = 1, 2, \ldots$. Prove or disprove whether $X_T \longrightarrow \mathbb{E}[X_T]$

(a) in mean square,

(b) almost surely,

(c) in probability.

Hint: You may use the converse of the Borel-Cantelli lemma:

$$\sum_{T=1}^{\infty} P\left(|X_T| \geq \epsilon\right) = \infty \Rightarrow P\left(\lim_{T \to \infty} |X_T| \geq \epsilon\right) = 1$$

for X_T independent.

Problem 6
Let $(y_1, \ldots, y_T)$ be a sample from a weakly stationary process with

$$\begin{aligned} \mathbb{E}\left[y_i\right] &= \mu \ \forall i \\ \operatorname{Cov}\left[y_i, y_{i-j}\right] &= \gamma_j \ \forall i, j \\ \sum_{j=0}^{\infty} |\gamma_j| &< \infty \ . \end{aligned}$$

Show that the bias and variance of $\bar{y}_T$ asymptotically converge to zero and hence,

$$\bar{y}_T = \frac{1}{T} \sum_{i=1}^{T} y_i \xrightarrow{m.s.} \mu \ .$$

Problem 7 * (7 Points)
Assume that Y_1, Y_2, ... is a sequence of independent $(K \times 1)$ random vectors with $\mathbb{E}[Y_i] = \mu$ and $\mathbb{V}[Y_i] = \Sigma$ positive definite.
Prove the WLLN: $\frac{1}{T} \sum_{i=1}^{T} Y_i \xrightarrow{p} \mu$ by using

(a) the inequality of Chebyshev,

(b) convergence in mean square.

Problem 8 * (6 Points)

(a) Show that OLS is a consistent estimator. State all assumptions that you rely upon.

(b) Describe three practical scenarios where consistency will fail and point out the assumption/s that is/are violated. Is there an inherent fault in sampling the data or in the data-generating process? Can you think of strategies to avoid an inconsistent estimator in such cases?

Problem 9 * (7 Points)
For this exercise use the file `data.csv` and an appropriate software. The parameter values used for generating the data are $\beta^{\mathsf{T}} = (10, 0.4, 0.6)$ and $\sigma_\epsilon^2 = 1$.

(a) Read the csv file into you programme. There are $1e6$ observations. In the following, split them into 1000 samples with 100 observations each and discard the rest.

(b) Obtain the parameter estimates, b and $\hat{\sigma}_\epsilon^2$ by estimating with OLS the equation

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \epsilon_t. \tag{1}$$

and compute R^2 for each sample. Show the range of these estimates (min, max).

(c) Compute the means of the parameter estimates, $\bar{b}$ and $\bar{\hat{\sigma}}_\epsilon^2$ and compare them with the true values. Why are they similar/different?

(d) Plot a histogram (you can choose an appropriate bin size) of the four parameter estimates in b and $\hat{\sigma}_\epsilon^2$. What do you observe?

(e) Now increase the number of observations T to 1000 by using all observations in `data.csv`. Obtain the parameter estimates, b and $\hat{\sigma}_\epsilon^2$ for each sample. Compute the means, $\bar{b}$ and $\bar{\hat{\sigma}}_\epsilon^2$. Do you observe a change when comparing the results to those in part (c)? Why/why not? What do you think will happen if T is further increased? Explain you answer.

(f) Plot a histogram (you can choose an appropriate bin size) of the four parameter estimates in b and $\hat{\sigma}_\epsilon^2$. What do you observe compared with part (d)?

(g) Plot a histogram (you can choose an appropriate bin size) of $\sqrt{T}(b_i - \beta_i), i = 1, 2, 3$ and $\sqrt{T}(\hat{\sigma}_\epsilon^2 - \sigma_\epsilon^2)$. What distribution does this resemble?